

Cost Function for CES Production Functions

For $w, r > 0$ and $Q, K, L > 0$, consider a constant-returns-to-scale production function:

$$Q^\rho = K^\rho + L^\rho \quad \rho \in (-\infty, 1) \setminus \{0\} \quad \text{in other terms:}$$

$$Q(L, K) = (K^\rho + L^\rho)^{\frac{1}{\rho}} \quad \rho \in (-\infty, 1) \setminus \{0\}$$

We are interested in the cost function associated with this production technology, which is the solution to the following optimization problem.

$$\begin{aligned} c(Q; w, r) &= \min_{K, L} wL + rK \quad \text{s.t.} \quad K > 0, L > 0, \quad Q^\rho = K^\rho + L^\rho \\ &= \min_{K, L} wL + rK \quad \text{s.t.} \quad K > 0, L > 0, \quad Q = (K^\rho + L^\rho)^{\frac{1}{\rho}} \end{aligned}$$

We could solve this using KKT optimization, but since the function is CRS, there is an easier way. CRS functions are homogeneous of degree one and thus have cost functions of the following form.

$$c(Q; w, r) = Qc(1; w, r)$$

Therefore, their marginal cost and average cost are identical.

$$c(1; w, r) = MC = AC = \frac{rK^*(Q) + wL^*(Q)}{Q}$$

Now consider the marginal and average products of our inputs.

$$\begin{aligned} MP_L &= \frac{\partial Q}{\partial L} = (K^\rho + L^\rho)^{\frac{1-\rho}{\rho}} L^{\rho-1} & AP_L &= \frac{Q}{L^*(Q)} \\ MP_K &= \frac{\partial Q}{\partial K} = (K^\rho + L^\rho)^{\frac{1-\rho}{\rho}} K^{\rho-1} & AP_K &= \frac{Q}{K^*(Q)} \end{aligned}$$

For $Q > 0$ and positive input prices, the cost-minimizing input bundle is interior. For $\rho < 0$, positive output requires both inputs. For $\rho \in (0, 1)$, the marginal product of an input becomes infinite as that input approaches zero, so a zero-input corner cannot be optimal at finite positive prices. At an interior solution, we have the following:

$$\frac{w}{r} = \frac{MP_L}{MP_K} = \frac{(K^\rho + L^\rho)^{\frac{1-\rho}{\rho}} L^{\rho-1}}{(K^\rho + L^\rho)^{\frac{1-\rho}{\rho}} K^{\rho-1}} = \left(\frac{L}{K}\right)^{\rho-1}$$

Let $\sigma = \frac{1}{1-\rho}$. At this interior solution, we can describe L as a function of K and thus both K and L as functions of Q .

$$\begin{aligned} L(K) &= \left(\frac{w}{r}\right)^{\frac{1}{\rho-1}} K = \left(\frac{w}{r}\right)^{-\sigma} K, \\ K(Q) &= Q \left(1 + \left(\frac{w}{r}\right)^{-\rho\sigma}\right)^{-\frac{1}{\rho}}, \\ L(Q) &= Q \left(\frac{w}{r}\right)^{-\sigma} \left(1 + \left(\frac{w}{r}\right)^{-\rho\sigma}\right)^{-\frac{1}{\rho}}. \end{aligned}$$

Substituting these conditional factor demands into total cost and simplifying gives the CES unit cost:

$$c(1; w, r) = (w^{1-\sigma} + r^{1-\sigma})^{\frac{1}{1-\sigma}}.$$

Since the technology is CRS, the cost function is

$$c(Q; w, r) = Q (w^{1-\sigma} + r^{1-\sigma})^{\frac{1}{1-\sigma}}.$$

Equivalently, by Shephard's lemma, the conditional factor demands are

$$\begin{aligned} L^*(Q; w, r) &= Q (w^{1-\sigma} + r^{1-\sigma})^{\frac{\sigma}{1-\sigma}} w^{-\sigma}, \\ K^*(Q; w, r) &= Q (w^{1-\sigma} + r^{1-\sigma})^{\frac{\sigma}{1-\sigma}} r^{-\sigma}. \end{aligned}$$

These demands satisfy production and tangency. For $\rho < 1$, production is concave, so $\{(K, L) : Q(K, L) \geq Q\}$ is convex; tangency therefore gives a global cost minimum.