

# Handout 2 - ECON703 (Fall 2023)

## 1 Supremum and Infimum

**Example 1.1** (Approaching One). Let  $X = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$ . Then  $\sup X = 1$ .

*Proof. Step 1:* Show that 1 is an upper bound of  $X$ .  
Observe that  $\forall n \in \mathbb{N} : \frac{1}{n} > 0$ . Thus  $\forall n \in \mathbb{N} : 1 - \frac{1}{n} \leq 1$ .

**Step 2:** Show that for any upper bound  $u$  of  $X$ , we have  $u \geq 1$ .  
Let  $u$  be an upper bound of  $X$  and assume (for the sake of contradiction) that  $u < 1$ . But then:

$$\exists k \in \mathbb{N} \quad \text{s.t.} \quad \frac{1}{k} < 1 - u \implies \exists k \in \mathbb{N} \quad \text{s.t.} \quad 1 - \frac{1}{k} > u \implies \exists x \in X \quad \text{s.t.} \quad x > u$$

**Contradiction!** Thus, 1 is the supremum of  $X$ . □

**Example 1.2** (Supremum and Infimum of a Square). Let  $A = (0, 1) \times (0, 1) \subset \mathbb{R}^2$  and consider the following partial order  $\preceq$  on  $\mathbb{R}^2$ :

$$\forall a, b \in A : (a \preceq b) \iff (b_1 \geq a_1 \wedge b_2 \geq a_2)$$

Then

$$\sup A = (1, 1) \quad \text{and} \quad \inf A = (0, 0)$$

*Proof.* We only consider the supremum; the infimum works analogously.

**Step 1:** Show that  $(1, 1)$  is an upper bound of  $A$  with respect to the partial order.

$$\forall a \in A : a_1 < 1 \wedge a_2 < 1 \implies \forall a \in A : a \preceq (1, 1)$$

Therefore,  $(1, 1)$  is an upper bound of  $A$  with respect to the partial order  $\preceq$ .

**Step 2:** Show that for any other upper bound  $u$  of  $A$ , we have  $(1, 1) \preceq u$ . Let  $u$  be an upper bound of  $A$ . Then:

$$\begin{aligned} \forall a \in A : a \preceq u &\implies \forall a \in A : u_1 \geq a_1 \wedge u_2 \geq a_2 \\ &\implies \forall x \in (0, 1) : u_1 \geq x \wedge u_2 \geq x \\ &\implies u_1 \geq 1 \wedge u_2 \geq 1 \\ &\implies (1, 1) \preceq u \end{aligned}$$

Thus,  $(1, 1)$  is the supremum of  $A$ . □

## 2 Metric Spaces and Open/Closed Sets

**Example 2.1** (Euclidean Space with Maximum Distance). Let  $n \in \mathbb{N}$ . Then  $\mathbb{R}^n$  combined with the following operations is a vector space over the real numbers  $\mathbb{R}$ .

$$\begin{aligned} +_{\mathbb{V}} : \mathbb{R}^n \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \quad \forall x, y \in \mathbb{R}^n \quad (x +_{\mathbb{V}} y)_i = x_i + y_i \quad \forall i = 1, \dots, n \\ \cdot_{\mathbb{V}} : \mathbb{R} \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \quad \forall x \in \mathbb{R}^n \forall \lambda \in \mathbb{R} \quad (\lambda \cdot_{\mathbb{V}} x)_i = \lambda x_i \quad \forall i = 1, \dots, n \end{aligned}$$

We call it  $n$ -dimensional Euclidean space. The maximum distance on this space is

$$d(x, y) = \max_{i=1, \dots, n} |x_i - y_i|$$

Together, these form a metric space.

*Proof.* To show that this is a metric space, we have to show that  $d$  satisfies the properties of a metric.

1.  $\forall v \in \mathbb{V} : d(v, v) = 0$
2. **Positivity:**  $\forall u, v \in \mathbb{V}$  with  $u \neq v : d(u, v) > 0$
3. **Symmetry:**  $\forall u, v \in \mathbb{V} : d(u, v) = d(v, u)$
4. **Triangle Inequality:**  $\forall u, v, w \in \mathbb{V} : d(u, w) \leq d(u, v) + d(v, w)$

We check them one by one.

1. Holds since  $\forall v \in \mathbb{R}^n \forall i = 1, \dots, n : v_i - v_i = 0 \quad \checkmark$
2. Holds since  $u \neq v$  implies  $\exists i \in \{1, \dots, n\}$  such that  $|u_i - v_i| > 0 \quad \checkmark$
3. Holds since  $\forall a, b \in \mathbb{R} : |a - b| = |b - a| \quad \checkmark$
4. We check this one in a bit more detail. Let  $x, y, z \in \mathbb{R}^n$

$$\begin{aligned} d(x, z) &= \max_{i=1, \dots, n} |x_i - z_i| = \max_{i=1, \dots, n} |x_i - y_i + y_i - z_i| \\ &\leq \max_{i=1, \dots, n} [ |x_i - y_i| + |y_i - z_i| ] \\ &\leq \max_{i=1, \dots, n} |x_i - y_i| + \max_{j=1, \dots, n} |y_j - z_j| = d(x, y) + d(y, z) \quad \checkmark \end{aligned}$$

□

One important thing to understand about open and closed sets is the following:

A subset is not like a door: it can be open, closed, both, or neither.

**Example 2.2** (Open and Closed Sets). Consider the following sets as subsets of  $\mathbb{R}^2$  and let  $d(x, y)$  be the Euclidean distance between two points  $x, y \in \mathbb{R}^2$ .

- |  |   |
|--|---|
| 1. $\mathbb{R}^2$  | 6. $D = A \setminus \{x \in \mathbb{R}^2 \mid x_1 > 0\}$    |
| 2. $\emptyset$   | 7. $E = B \setminus \{x \in \mathbb{R}^2 \mid x_1 > 0\}$    |
| 3. $A = \{x \in \mathbb{R}^2 \mid d(x, (0, 0)) \leq 1\}$ | 8. $F = A \setminus \{x \in \mathbb{R}^2 \mid x_1 \geq 0\}$ |
| 4. $B = \{x \in \mathbb{R}^2 \mid d(x, (0, 0)) < 1\}$    | 9. $G = B \setminus \{x \in \mathbb{R}^2 \mid x_1 \geq 0\}$ |
| 5. $C = \{x \in \mathbb{R}^2 \mid d(x, (0, 0)) = 1\}$    | 10. $H = \{(0, 0)\}$  |

Are they open, closed, both, or neither?

### 3 Sequences and Series

**Definition 3.1** (Convergent Sequence). Let  $(X, d_X)$  be a metric space and  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . We say that  $x_n$  converges to  $\bar{x}$ , denoted by  $\lim_{n \rightarrow \infty} x_n = \bar{x}$  or  $x_n \rightarrow \bar{x}$  as  $n \rightarrow \infty$ , if

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n > N_\epsilon : d_X(x_n, \bar{x}) < \epsilon$$

**Example 3.1** (Geometric Series). Let  $(x_n)_{n \in \mathbb{N}}$  be given by  $x_n = \sum_{i=0}^{n-1} r^i$  for  $r \in (0, 1)$ . Then  $(x_n)_{n \in \mathbb{N}}$  is a convergent sequence with  $\lim_{n \rightarrow \infty} x_n = \frac{1}{1-r}$ .

*Proof.* First, we show that the following formula gives the value of a partial sum of the geometric series.

$$x_n = \sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r}$$

We will do so using a proof by induction.

- **Base Case:**  $n = 1$

$$x_1 = \sum_{i=0}^0 r^i = r^0 = 1 = \frac{1 - r^1}{1 - r}$$

- **Induction Step:** Assume that  $x_n = \frac{1 - r^n}{1 - r}$ . Then

$$x_{n+1} = x_n + r^n = \frac{1 - r^n}{1 - r} + r^n = \frac{(1 - r^n) + (1 - r)r^n}{1 - r} = \frac{(1 - r^n) + (r^n - r^{n+1})}{1 - r} = \frac{1 - r^{n+1}}{1 - r}$$

Thus, we have shown that the assumed formula for the partial sums holds.

If the limit exists, this gives us the following statement.

$$\sum_{i=0}^{\infty} r^i = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} r^i = \lim_{n \rightarrow \infty} \frac{1 - r^n}{1 - r}$$

We can show that the second limit exists in the following way.

$$\lim_{n \rightarrow \infty} \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} \left( 1 - \lim_{n \rightarrow \infty} r^n \right) = \frac{1}{1 - r} (1 - 0) = \frac{1}{1 - r}$$

where the second equality holds since  $|r| < 1$ . Thus, the series is convergent and converges to  $\frac{1}{1-r}$ . □

**Example 3.2** (Harmonic Series). Let  $(x_n)_{n \in \mathbb{N}}$  be given by  $x_n = \sum_{i=1}^n \frac{1}{i}$ . Then  $(x_n)_{n \in \mathbb{N}}$  diverges.

*Proof.* Observe that we can group the terms of the harmonic series as follows:

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \sum_{i=1}^{\infty} \left( \sum_{j=1}^{2^{i-1}} \frac{1}{2^{i-1} + j} \right) = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \dots$$

But then

$$\forall i \in \mathbb{N} : \sum_{j=1}^{2^{i-1}} \frac{1}{2^{i-1} + j} \geq \frac{1}{2}$$

and thus

$$\sum_{i=1}^{\infty} \frac{1}{i} \geq \sum_{i=1}^{\infty} \frac{1}{2} = \infty$$

Thus  $\sum_{i=1}^{\infty} \frac{1}{i}$  diverges. □

## 4 Continuity

**Definition 4.1** (Continuity). Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and  $f : X \rightarrow Y$  a function from  $X$  to  $Y$ . We say that  $f$  is continuous at a point  $x \in X$  if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall x' \in X : d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon$$