# Handout 3 - ECON703 (Fall 2023)

## 1 Cauchy Sequences and Completeness

**Definition 1.1** (Cauchy-Sequence)**.** *Let $(X, d_X)$ be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X$. Then $(x_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence if*

$$\forall\epsilon > 0\, \exists N_\epsilon \in \mathbb{N} \quad \textit{such that} \quad \forall m, n > N_\epsilon : \, d_X(x_n, x_m) < \epsilon$$

**Example 1.1** (Partial Sums of the Geometric Series)**.** *The partial sums of the geometric series for $r = \frac{1}{2}$ are a Cauchy sequence.*

*Proof.* Recall from Handout 2 that for $|r| < 1$

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r} \quad \text{for } r = \frac{1}{2} \quad \sum_{i=0}^{n} 2^{-i} = 2(1 - 2^{-n-1}) = 2 - 2^{-n}$$

Then, we can find the following formula for the distance between two partial sums. Without loss of generality, let $m > n$.

$$0 < |\sum_{i=0}^{m} r^i - \sum_{i=0}^{n} r^i| = |2 - 2^{-m} - 2 + 2^{-n}| = |2^{-n} - 2^{-m}| = 2^{-n} - 2^{-m} < 2^{-n}$$

Choose some $\epsilon > 0$, then $\exists N_\epsilon \in \mathbb{N}$ such that $2^{-N_\epsilon} < \epsilon$. And $\forall n > N_\epsilon : \, 2^{-n} < \epsilon$. But then:

$$\forall m, n > N_\epsilon : \quad |\sum_{i=0}^{m} r^i - \sum_{i=0}^{n} r^i| < \epsilon$$

□

**Definition 1.2** (Complete Metric Space)**.** *Let $(X, d_X)$ be a metric space. $(X, d_X)$ is called complete if every Cauchy sequence in $(X, d_X)$ converges in $(X, d_X)$, i.e. has a limit in $X$.*

**Theorem 1.1** (The Real Numbers)**.** *Consider the metric space $(\mathbb{R}, d)$ where $d(x, y) = |y - x|$. This metric space is complete.*

*Proof.* Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is bounded because

$$\exists N \in \mathbb{N}\, \forall m > N : \quad d(x_{N+1}, x_m) < 1$$

$\implies \{x_n\}_{n>N}$ is bounded below by $x_{N+1} - 1$ and above by $x_{N+1} + 1$. Thus $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence and contains a convergent subsequence $(x_{n_i})_{i\in\mathbb{N}}$ (Bolzano-Weierstrass Theorem) with $\lim_{i\to\infty} x_{n_i} = x^*$. Choose $\epsilon > 0$ arbitrary.

$$\exists N_{\epsilon,1} \in \mathbb{N}\, \forall n, m > N_{\epsilon,1} : \quad d(x_n, x_m) < \frac{\epsilon}{2} \quad \text{and} \quad \exists N_{\epsilon,2} \in \mathbb{N}\, \forall k > N_{\epsilon,2} : \quad d(x_{n_k}, x^*) < \frac{\epsilon}{2}$$

Let $N_\epsilon = \max\{N_{\epsilon,1}, N_{\epsilon,2}\}$. Then

$$\forall k > N_\epsilon : d(x_k, x^*) \overset{\Delta-\text{ineq.}}{\leq} d(x_k, x_{n_k}) + d(x_{n_k}, x^*) \leq \epsilon$$

Thus $\lim_{n\to\infty} x_n = x^*$. □

## 2 Bolzano-Weierstrass Theorem

**Theorem 2.1** (Bolzano-Weierstrass in $\mathbb{R}$)**.** *Every infinite bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ has a convergent subsequence.*

*Intuitive "Proof".* Call $n \in \mathbb{N}$ a peak of $(x_n)_{n\in\mathbb{N}}$ if $m > n \implies x_n > x_m$.
Consider the following cases:

1. $(x_n)_{n\in\mathbb{N}}$ has infinitely many peaks $n_1 < n_2 < n_3 < \ldots$.
   Then $(x_{n_j})_{j\in\mathbb{N}}$ is monotonically decreasing and bounded below. It is therefore convergent by the monotone convergence theorem.
2. $(x_n)_{n\in\mathbb{N}}$ has finitely many peaks. Let $N$ be the last peak and let $n_1 = N + 1$.
   $n_1$ is not a peak $\implies \exists n_2 > n_1$ such that $x_{n_2} \geq x_{n_1}$.
   $n_2$ is not a peak $\implies \exists n_3 > n_2$ such that $x_{n_3} \geq x_{n_2}$ and so forth.
   Then $(x_{n_j})_{j\in\mathbb{N}}$ is a bounded monotonically increasing sequence. It is convergent by the bounded convergence theorem.
3. $(x_n)_{n\in\mathbb{N}}$ has no peaks. Let $N = -1$. The argument provided in 2 applies.

□

**Theorem 2.2** (Bolzano-Weierstrass)**.** *Every infinite bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^n$ has a convergent subsequence.*

*Proof.* We repeatedly apply Theorem 2.1 to the dimensions of $(x_n)_{n\in\mathbb{N}}$ to construct such a sequence.

- The first dimension of $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Apply Theorem 2.1 to the first dimension of $(x_n)_{n\in\mathbb{N}}$ to obtain a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ whose first coordinate converges.
- The second dimension of $(x_{n_i})_{i\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Apply Theorem 2.1 to the second dimension of $(x_{n_i})_{i\in\mathbb{N}}$ to obtain a sequence $(x_k)_{k\in\mathbb{N}}$ whose first and second coordinates converge. $(x_k)_{k\in\mathbb{N}}$ is a subsequence of $(x_{n_i})_{i\in\mathbb{N}}$ and thus a subsequence of $(x_n)_{n\in\mathbb{N}}$.
- ...

Iterate this process to obtain a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges in all of its $n$ dimensions. This sequence is a convergent subsequence of $(x_n)_{n\in\mathbb{N}}$. □

## 3 Contraction Mappings

**Definition 3.1** (Contraction Mapping)**.** *Let $(X, d_X)$ be a metric space. A function $f : X \to X$ is called a contraction mapping on $X$ if*

$$\exists \beta \in [0,1) \quad \textit{such that} \quad \forall x, y \in X : \quad d_X(f(x), f(y)) \leq \beta d_X(x, y)$$

**Example 3.1** (Example)**.** *A simple example of a contraction mapping is $f(x) = a + bx$ for $b \in [0,1)$.*

**Example 3.2** (Counterexample)**.** *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given by $f(x) = x\left(1 - \frac{1}{1+x}\right)$.*

*Observe that this function shrinks distances between points. Wlog[1], let $y > x$*

$$|f(y) - f(x)| = y\left(1 - \frac{1}{1+y}\right) - x\left(1 - \frac{1}{1+x}\right) < y - x = |y - x|$$
$$\iff \frac{x}{1+x} < \frac{y}{1+y} \iff x(1+y) < y(1+x) \iff x < y \quad \checkmark$$

[1] without loss of generality

*Then let $y = x + 1$ and thus $|y - x| = 1$*

$$\lim_{x\to\infty} |f(y) - f(x)| = \lim_{x\to\infty} |y\left(1 - \frac{1}{1+y}\right) - x\left(1 - \frac{1}{1+x}\right)|$$

$$= \lim_{x\to\infty} (x+1)\left(1 - \frac{1}{2+x}\right) - x\left(1 - \frac{1}{1+x}\right) = \lim_{x\to\infty} \frac{1}{x+2} - \frac{1}{x+1} + 1 = 1$$

*Even though $f$ shrinks distances, its Lipschitz constant is not bounded away from one. Thus, it is not a contraction mapping.*

**Definition 3.2** (Fixed Point)**.** *Let $f : X \to X$ be a function from $X$ to itself. We say $x \in X$ is a fixed point of $f$ if $f(x) = x$.*

**Theorem 3.1** (Banach Fixed Point Theorem)**.** *Let $(X, d_X)$ be a non-empty complete metric space. Let $f : X \to X$ be a contraction mapping on $X$. Then $f$ has a unique fixed point $x^*$ in $X$.*

*Proof.* Choose $x_0 \in X$ arbitrarily. Define the sequence $(x_n)_{n\in\mathbb{N}}$ recursively by: $\forall n \in \mathbb{N}_0 : \; x_{n+1} = f(x_n)$. Note the following statement, which follows from iterated application of the contraction mapping definition.

$$d_X(x_{n+1}, x_n) \leq \beta^n d_X(x_1, x_0)$$

Next, we show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ and, without loss of generality, let $m > n$

$$d_X(x_m, x_n) \overset{\Delta-\text{ineq.}}{\leq} \sum_{i=n}^{m-1} d_X(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \beta^i d_X(x_0, x_1) = \beta^n d_X(x_0, x_1) \sum_{i=0}^{m-n-1} \beta^i$$

$$\leq \beta^n d_X(x_0, x_1) \sum_{i=0}^{\infty} \beta^i = \beta^n d_X(x_0, x_1) \frac{1}{1-\beta}$$

Choose $\epsilon > 0$ arbitrary. Then

$$\exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \beta^{N_\epsilon} \frac{d_X(x_0, x_1)}{1-\beta} < \epsilon$$

and thus

$$\forall m, n > N_\epsilon : \quad d_X(x_m, x_n) < \epsilon$$

Therefore, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $(X, d_X)$ is complete, $(x_n)_{n\in\mathbb{N}}$ converges in $(X, d_X)$.

Let $x^* = \lim_{n\to\infty} x_n$.

$$d_X(x^*, f(x^*)) \leq d_X(x^*, x_m) + d_X(x_m, f(x^*)) \leq d_X(x^*, x_m) + \beta d_X(x_{m-1}, x^*)$$

But

$$\lim_{m\to\infty} d_X(x^*, x_m) = 0$$

Thus, we can bound $d_X(x^*, f(x^*))$ by arbitrarily small positive values. Thus, $d_X(x^*, f(x^*)) = 0$ meaning that $x^*$ is a fixed point of $f$.

Assume (for the sake of contradiction) that there is a second fixed point; let's call it $x^{**}$.

$$d_X(x^*, x^{**}) = d_X(f(x^*), f(x^{**})) \leq \beta d_X(x^*, x^{**})$$

Since $\beta < 1$, this implies $d_X(x^*, x^{**}) = 0$ and therefore $x^* = x^{**}$. Thus $x^*$ must be the unique fixed point of $f$. □

**Example 3.3** (Dynamic Programming)**. Don't worry!**

*You'll see this in much more detail during your macroeconomics class! So I will leave out all the nuance here. Let $C_b(\mathbb{R})$ be the space of continuous and bounded functions on the real numbers. Let $T : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ be defined pointwise by:*

$$T[v](x) = \max_{z\in\Gamma(x)} \left(u(x-z) + \beta v(z)\right)$$

*where $\Gamma$ is some (feasibility)-correspondence.*

*Then we can show (under some conditions) that $T$ is a contraction mapping on $C_b(\mathbb{R})$. It thus has a fixed point, which will be useful in* **A LOT** *of economic problems. (Dynamic Optimization)*

# 4 Convergence of Functions

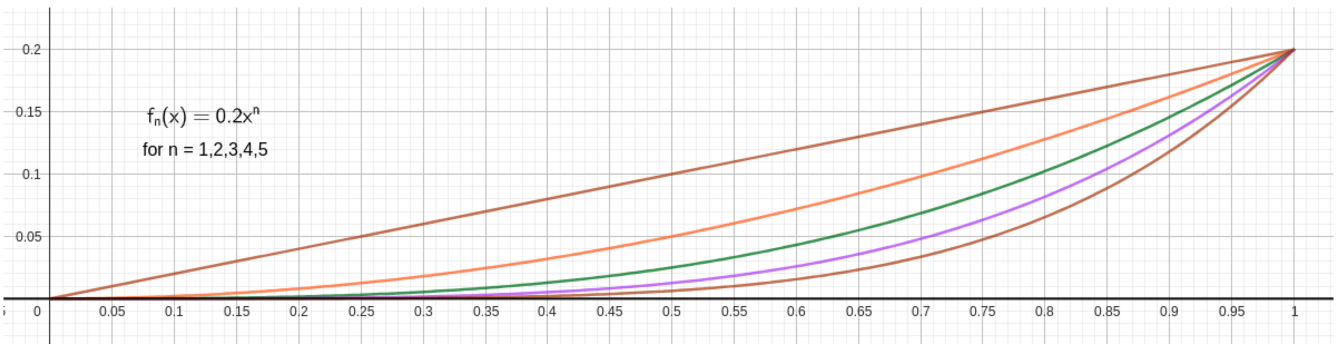
**Definition 4.1** (Pointwise Convergence). *Let $(f_n)_{n\in\mathbb{N}}$ with $f : \mathbb{R} \to \mathbb{R}$ be a sequence of functions. We say that $f_n$ converges pointwise to some function $f : \mathbb{R} \to \mathbb{R}$ if:*

$$\forall x \in \mathbb{R} : \quad \lim_{n\to\infty} f_n(x) = f(x)$$

**Definition 4.2** (Uniform Convergence). *Let $(f_n)_{n\in\mathbb{N}}$ with $f : \mathbb{R} \to \mathbb{R}$ be a sequence of functions. We say that $f_n$ converges uniformly to some function $f : \mathbb{R} \to \mathbb{R}$ if:*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = 0$$

**Example 4.1** (Pointwise but not Uniform Convergence). *Let $f_n : [0, 1) \to [0, 1)$ be given by $f_n(x) = x^n$. Then $f_n$ converges pointwise to $f(x) = 0$. However, $f_n$ does not converge uniformly.*




*Proof.* Take $x \in [0, 1)$ arbitrarily. Then $\lim_{n\to\infty} x^n = 0$. Thus $f_n$ converges pointwise to $f(x) = 0$.

$f$ is the only candidate for uniform convergence. However,

$$\forall n \in \mathbb{N} : \quad \sup_{x\in[0,1)} |x^n - 0| = 1$$

Thus, $f_n$ does not converge uniformly. □