Handout 4 - ECON703 (Fall 2023)

1 Cluster Points and the Squeeze Theorem

Definition 1.1 (Cluster Point / Accumulation Point). *Let (X, d_X) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $y \in X$ is a cluster point of $(x_n)_{n\in\mathbb{N}}$ if $(x_n)_{n\in\mathbb{N}}$ has a subsequence converging to y.*

Theorem 1.1 (Supremum of the Set of Cluster Points). *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded[1] sequence in $\mathbb{R}$ and $\mathbb{Y} \subset \mathbb{R}$ be the set of cluster points of $(x_n)_{n\in\mathbb{N}}$. Then $\sup \mathbb{Y} = \limsup_{n\to\infty} x_n$.*

Proof. Recall the definition of the limit superior:

$$b_n = \sup\{x_k \,|k \geq n\} \quad \text{i.e.} \quad \lim_{n\to\infty} b_n = \limsup_{n\to\infty} x_n$$

Then, as an exercise, one can show that

$$\exists (x_{n_i})_{i\in\mathbb{N}} \quad \text{such that} \quad \lim_{i\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n$$

and thus

$$\limsup_{n\to\infty} x_n \in \mathbb{Y} \quad \text{which gives us} \quad \sup \mathbb{Y} \geq \limsup_{n\to\infty} x_n$$

Let $(x_{n_k})_{k\in\mathbb{N}}$ be a convergent subsequence of $(x_n)_{n\in\mathbb{N}}$. Then we have $\lim_{k\to\infty} x_{n_k} = x^* \in \mathbb{Y}$. Then

$$\forall k \in \mathbb{N} : \; x_{n_k} \leq b_{n_k}$$

Passing to the limit gives:

$$x^* = \lim_{k\to\infty} x_{n_k} \leq \lim_{k\to\infty} b_{n_k} = \limsup_{n\to\infty} x_n$$

and thus $\forall y \in \mathbb{Y} : \; y \leq \limsup_{n\to\infty} x_n$. □

Analogously, this applies to the limit inferior of this sequence as a lower bound of $\mathbb{Y}$.

Theorem 1.2 (Squeeze Theorem / Sandwich Theorem). *Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ and $h : [a, b] \to \mathbb{R}$ and $c \in [a, b]$. We can also make this work if f, g and h are only defined on $[a, b]\backslash\{c\}$. Assume that*

$$\forall x \in [a, b]\backslash\{c\} : \quad g(x) \leq f(x) \leq h(x)$$

and

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

Then $\lim_{x\to c} f(x) = L$.

Proof.

$$\begin{aligned} L = \lim_{x\to c} g(x) \leq \liminf_{x\to c} f(x) \leq \limsup_{x\to c} f(x) \leq \lim_{x\to c} h(x) = L \\ \implies \liminf_{x\to c} f(x) = \limsup_{x\to c} f(x) = L \\ \implies \lim_{x\to c} f(x) = L \end{aligned}$$

The second implication follows from Homework Problem 14. □

[1] We can make this work with unbounded sequences, but then $\limsup_{n\to\infty} x_n = \infty$ becomes a possibility.

Example 1.1 (Strange Sine Function)**.** *Consider the function* $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *with* $f(x) = x^2 \sin(\frac{1}{x})$*. Then since* $\sin(x) \in [-1, 1]$*, we have:*

$$-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$$

Then we can find the following limits:

$$\lim_{x\to 0} -x^2 = 0 \quad \textit{and} \quad \lim_{x\to 0} x^2 = 0$$

so the squeeze theorem implies that $\lim_{x\to 0} f(x) = 0$*.*

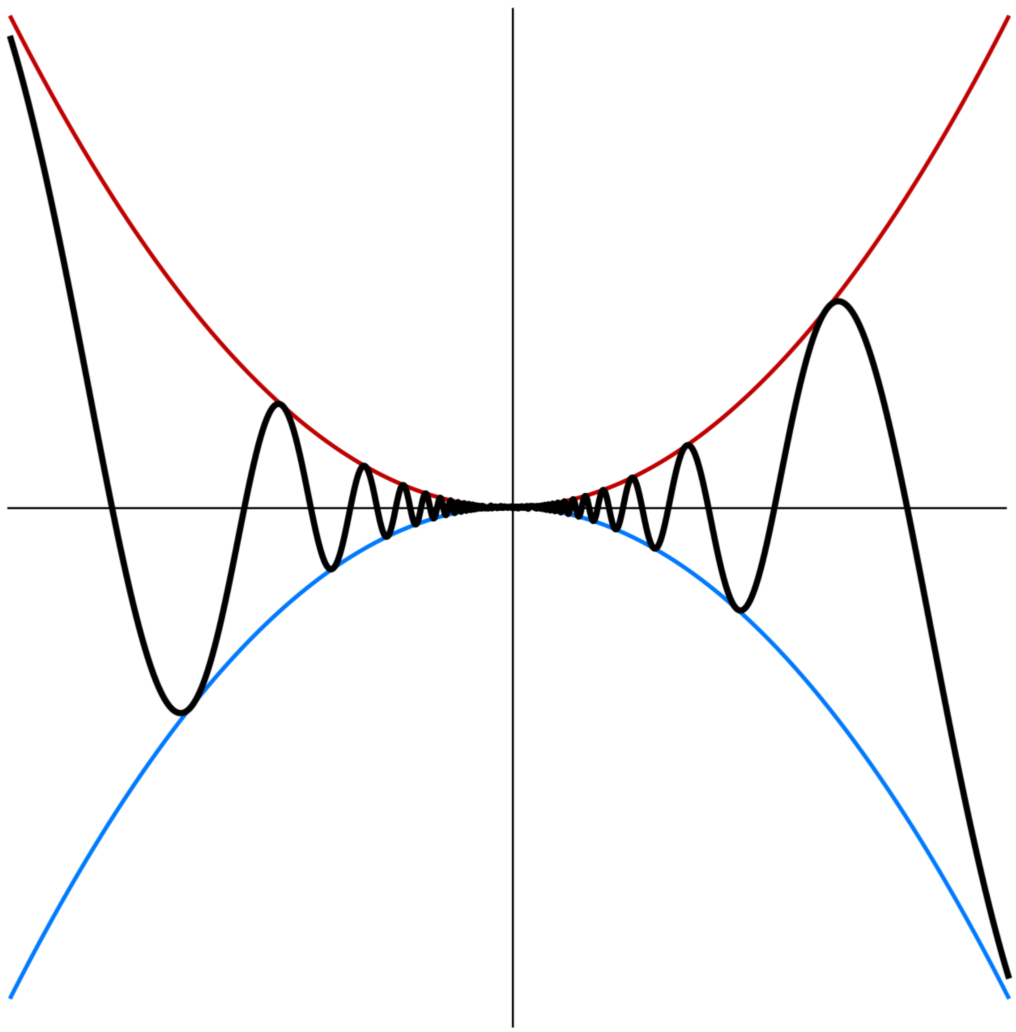

Figure 1: Source: `https://en.wikipedia.org/wiki/File:Inst_satsen.png`

2 Lipschitz-Continuity

Definition 2.1 (Lipschitz-Continuity)**.** *Let* (X, d_X) *and* (Y, d_Y) *be two metric spaces. We say a function* $f : X \to Y$ *is Lipschitz continuous if*

$$\exists K \in \mathbb{R}_{\geq 0} \quad \textit{such that} \quad \forall x, x' \in X : \, d_Y(f(x), f(x')) \leq K d_X(x, x')$$

We call any such K *a Lipschitz constant of* f *and the smallest such* K *the dilation of* f*.*

Example 2.1 (Absolute Value)**.** *The function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = |x|$ *is Lipschitz continuous with dilation 1.*

Theorem 2.1 (Lipschitz-Continuous Functions are Continuous). *Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $f : X \to Y$ be a Lipschitz continuous function with dilation $K \in \mathbb{R}$. Then f is continuous.*

Proof. Since f is Lipschitz continuous, we know that

$$\forall x, x' \in X : \; d_Y(f(x), f(x')) \le K d_X(x, x')$$

If $K = 0$, then $d_Y(f(x), f(x')) = 0$ for all $x, x' \in X$ and continuity is immediate. Now consider the case $K > 0$. Choose $\epsilon > 0$ arbitrary and set $\delta = \frac{\epsilon}{K}$. Then

$$d_X(x, x') < \delta \implies d_X(x, x') < \frac{\epsilon}{K} \implies K d_X(x, x') < \epsilon \implies d_Y(f(x), f(x')) < \epsilon$$

Thus, f is continuous. □

3 Continuity

Theorem 3.1 (Sequence Definition of Continuity). *Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$. f is continuous at $x \in X$ if and only if*

$$\forall (x_n)_{n\in\mathbb{N}} \quad \textit{with} \quad \lim_{n\to\infty} x_n = x : \quad \lim_{n\to\infty} f(x_n) = f(x)$$

Proof[2]. **Step 1**: if this property holds, f is continuous at x

We show the contrapositive, i.e., if the epsilon-delta definition does not hold at x, then the above sequence property is violated. Thus, assume that f is not continuous at x, i.e.

$$\exists \epsilon > 0 \quad \text{such that} \quad \forall \delta > 0 \, \exists x' \in X \quad \text{with} \quad d_X(x', x) < \delta \, \wedge \, d_Y(f(x'), f(x)) \ge \epsilon$$

Let ϵ_0 be the $\epsilon > 0$ at which our continuity definition is violated. Now consider the following sequences

$$(\delta_i)_{i\in\mathbb{N}} \quad \text{with} \quad \delta_i = 2^{-i}$$

and

$$(x_i)_{i\in\mathbb{N}} \quad \text{with} \quad d_X(x, x_i) < \delta_i \, \wedge \, d_Y(f(x_i), f(x)) \ge \epsilon_0$$

But then

$$\lim_{i\to\infty} x_i = x \quad \text{and} \quad \lim_{i\to\infty} f(x_i) \ne f(x)$$

Thus, the property under investigation is violated.

Step 2: if f is continuous at x, this property holds

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $\lim_{n\to\infty} x_n = x$ and choose an arbitrary $\epsilon > 0$. Since f is continuous at x we have

$$\exists \delta > 0 \quad \text{such that} \quad \forall x' \in X \quad \text{with} \quad d_X(x, x') < \delta : \; d_Y(f(x), f(x')) < \epsilon$$

But since $\lim_{n\to\infty} x_n = x$, we have

$$\exists N_\delta \in \mathbb{N} \quad \text{such that} \quad \forall n > N_\delta : \; d_X(x_n, x) < \delta$$

and thus

$$\forall n > N_\delta : \; d_Y(f(x_n), f(x)) < \epsilon$$

But then

$$\lim_{n\to\infty} f(x_n) = f(x)$$

□