

# Handout 5 - ECON703 (Fall 2023)

## 1 Vector Spaces and Norms

In the following, whenever we use the term field ( $\mathbb{F}$ ), you can think of  $\mathbb{R}$  together with our usual addition and multiplication. We can define many of these objects more generally, but let's stay in the reals for now.

**Definition 1.1** (Vector Space). *A vector space  $(\mathbb{V}, +_{\mathbb{V}}, \cdot_{\mathbb{V}})$  over a field  $\mathbb{F}$  (with addition  $+_{\mathbb{F}}$  and multiplication  $\cdot_{\mathbb{F}}$ ) is a non-empty set  $\mathbb{V}$  together with two binary operations  $+_{\mathbb{V}} : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$  and  $\cdot_{\mathbb{V}} : \mathbb{F} \times \mathbb{V} \rightarrow \mathbb{V}$  such that the following properties hold.*

- *Associativity of Vector Addition:*

$$\forall u, v, w \in \mathbb{V} : u + (v + w) = (u + v) + w$$

- *Commutativity of Vector Addition:*

$$\forall u, v \in \mathbb{V} : u + v = v + u$$

- *Existence of an Identity Element of Vector Addition:*

$$\exists 0_{\mathbb{V}} \in \mathbb{V} \text{ s.t. } \forall v \in \mathbb{V} : v + 0_{\mathbb{V}} = v$$

- *Existence of Inverse Elements of Vector Addition:*

$$\forall v \in \mathbb{V} \exists (-v) \in \mathbb{V} \text{ s.t. } v + (-v) = 0_{\mathbb{V}}$$

- *Compatibility of Scalar Multiplication with Field Multiplication:*

$$\forall v \in \mathbb{V} \forall \lambda, \mu \in \mathbb{F} : \lambda \cdot_{\mathbb{V}} (\mu \cdot_{\mathbb{V}} v) = (\lambda \cdot_{\mathbb{F}} \mu) \cdot_{\mathbb{V}} v$$

- *Existence of an Identity Element of Scalar Multiplication:*

$$\forall v \in \mathbb{V} : 1_{\mathbb{F}} \cdot_{\mathbb{V}} v = v$$

- *Distributivity of Scalar Multiplication w.r.t. Vector Addition:*

$$\forall u, v \in \mathbb{V} \forall \lambda \in \mathbb{F} : \lambda \cdot_{\mathbb{V}} (u +_{\mathbb{V}} v) = \lambda \cdot_{\mathbb{V}} u +_{\mathbb{V}} \lambda \cdot_{\mathbb{V}} v$$

- *Distributivity of Scalar Multiplication w.r.t. Field Addition:*

$$\forall v \in \mathbb{V} \forall \lambda, \mu \in \mathbb{F} : (\lambda +_{\mathbb{F}} \mu) \cdot_{\mathbb{V}} v = \lambda \cdot_{\mathbb{V}} v +_{\mathbb{V}} \mu \cdot_{\mathbb{V}} v$$

**Example 1.1** (Two-Dimensional Real Vectors). *The set of all ordered pairs of real numbers, i.e.,  $\mathbb{R}^2$  together with component-wise addition and scalar multiplication, is a vector space over  $\mathbb{R}$ .*

$$+ : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{defined by} \quad (x, y) + (a, b) = (x + a, y + b)$$

$$\cdot : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{defined by} \quad \lambda \cdot (x, y) = (\lambda x, \lambda y)$$

*Proof.* Check the properties...

□

**Definition 1.2** (Norm). A function  $\|\cdot\| : \mathbb{V} \rightarrow \mathbb{R}$  from a vector space  $\mathbb{V}$  to the underlying field  $\mathbb{R}$  is called a norm if it satisfies the following properties.

1. **Triangle Inequality:**

$$\forall u, v \in \mathbb{V} : \|u + v\| \leq \|u\| + \|v\|$$

2. **Absolute Homogeneity:**

$$\forall v \in \mathbb{V}, \lambda \in \mathbb{F} : \|\lambda v\| = |\lambda| \|v\|$$

3. **Positive Definiteness:**

$$\forall v \in \mathbb{V} : \|v\| \geq 0 \quad \text{and} \quad \|v\| = 0_{\mathbb{F}} \iff v = 0_{\mathbb{V}}$$

A function that satisfies properties 1 and 2 is called a seminorm.

**Example 1.2** ( $\ell_p$ -Norm). For  $p \in [1, \infty)$ , the following formula defines the  $\ell_p$ -norm.

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

This is a norm on an  $n$ -dimensional Euclidean space.

**Theorem 1.1** (Distance induced by a Norm). A norm  $\|\cdot\| : \mathbb{V} \rightarrow \mathbb{R}$  on a vector space  $(\mathbb{V}, +_{\mathbb{V}}, \cdot_{\mathbb{V}})$  induces a distance on  $(\mathbb{V}, +_{\mathbb{V}}, \cdot_{\mathbb{V}})$  as follows:

$$d(x, y) = \|x - y\|$$

*Proof.* Check the properties of a distance:

- $\forall v \in \mathbb{V} : d(v, v) = 0$  Holds by positive definiteness of the underlying norm ✓
- **Positivity:** Holds by positivity of the underlying norm ✓
- **Symmetry:** Holds by absolute homogeneity with  $\lambda = -1$  ✓
- **Triangle Inequality:** Let  $x, y, z \in \mathbb{V}$

$$d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y) \quad \checkmark$$

□

## 2 Inner Products and the Cauchy-Schwarz Inequality

**Definition 2.1** (Inner Product). An inner product on a vector space  $\mathbb{V}$  over  $\mathbb{R}$  is a function  $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$  satisfying the following properties.

- **Symmetry:**  $\forall v, u \in \mathbb{V} : \langle v, u \rangle = \langle u, v \rangle$
- **Linearity in the First Argument:**  $\forall u, v, w \in \mathbb{V} \forall \lambda, \mu \in \mathbb{F} : \langle \lambda v + \mu u, w \rangle = \lambda \langle v, w \rangle + \mu \langle u, w \rangle$
- **Positive Definiteness:**  $\forall v \in \mathbb{V} : (v \neq 0) \implies (\langle v, v \rangle > 0)$

**Example 2.1** (Canonical Inner Product on Euclidean Space). *The canonical inner product on  $\mathbb{R}^n$ ,  $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ , is given by*

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i$$

*Proof.* Check the properties of the inner product definition.

- **Symmetry:** Holds ✓
- **Linearity in the First Argument:** Holds ✓
- **Positive Definiteness:** Holds ✓

□

**Definition 2.2** (Inner Product Space). *An inner product space is a vector space  $\mathbb{V}$  over some field  $\mathbb{F}$  together with an inner product  $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{F}$ .*

**Theorem 2.1** (Cauchy-Schwarz Inequality). *Let  $(\mathbb{V}, \langle \cdot, \cdot \rangle)$  be an inner product space. Then*

$$\forall u, v \in \mathbb{V} : |\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle$$

*In Euclidean space with the standard inner product, the Cauchy-Schwarz inequality becomes:*

$$\left( \sum_{i=1}^n u_i v_i \right)^2 \leq \left( \sum_{i=1}^n u_i^2 \right) \left( \sum_{i=1}^n v_i^2 \right)$$

*with equality if and only if  $u$  and  $v$  are linearly dependent.*

*Proof.* Due to time constraints: see [https://en.wikipedia.org/wiki/Cauchy-Schwarz\\_inequality#Proofs](https://en.wikipedia.org/wiki/Cauchy-Schwarz_inequality#Proofs) □

### 3 The Space of Continuous and Bounded Functions

**Definition 3.1** (Uniformly Bounded Set of Functions). *Let  $X$  be an arbitrary set and  $\mathcal{F} = \{f_i : X \rightarrow \mathbb{R} \mid i \in \mathcal{I}\}$  be a set of functions indexed by some index set  $\mathcal{I}$ . We say that  $\mathcal{F}$  is bounded uniformly if*

$$\exists M \in \mathbb{R} \text{ such that } \forall i \in \mathcal{I} \forall x \in X : |f_i(x)| \leq M$$

**Definition 3.2** (Space of Continuous and Bounded Functions). *The space of continuous and bounded functions from a set  $X \subset \mathbb{R}^n$  to  $\mathbb{R}$ , denoted by  $\mathcal{C}_b(X)$ , is defined as the vector space consisting of the set*

$$\{f : X \rightarrow \mathbb{R} \mid f \text{ continuous and bounded}\}$$

*together with pointwise addition and scalar multiplication.*

$$\begin{aligned} + : \mathcal{C}_b(X) \times \mathcal{C}_b(X) &\rightarrow \mathcal{C}_b(X) & \text{ given by } & \forall f, g \in \mathcal{C}_b(X) \forall x \in X : (f + g)(x) = f(x) + g(x) \\ \cdot : \mathbb{R} \times \mathcal{C}_b(X) &\rightarrow \mathcal{C}_b(X) & \text{ given by } & \forall f \in \mathcal{C}_b(X) \forall \lambda \in \mathbb{R} \forall x \in X : (\lambda f)(x) = \lambda f(x) \end{aligned}$$

One important thing to note is that not all of these functions have to be bounded by the same bound  $M$ . In fact, if we assume that they are, this is not a vector space.

**Theorem 3.1** (The space of Continuous and Bounded Functions with the sup-norm is a Complete Metric Space).  *$\mathcal{C}_b(X)$  together with the operations defined above and the distance induced by the sup-norm*

$$d(f, g) = \sup_{x \in X} |g(x) - f(x)|$$

*is a complete metric space.*

*Proof.* Let  $f_i : X \rightarrow \mathbb{R}$  be a Cauchy sequence with respect to the sup-distance, i.e.

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n, m > N_\epsilon : \sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon$$

Then  $\forall x \in X$  the sequence  $(f_i(x))_{i \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}$  and thus convergent. This means that the pointwise limit of our function sequence exists. Let  $f : X \rightarrow \mathbb{R}$  be the pointwise limit of  $f_i$ , i.e.

$$\forall x \in X : \quad f(x) = \lim_{i \rightarrow \infty} f_i(x)$$

This will be our candidate for the limit of our function sequence in the sup-distance.

**Uniform Convergence:**

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n, m > N_\epsilon : \sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon$$

and thus, by linearity of the absolute value distance, we have

$$\forall x \in X : |f(x) - f_m(x)| = \left| \lim_{i \rightarrow \infty} f_i(x) - f_m(x) \right| = \lim_{i \rightarrow \infty} |f_i(x) - f_m(x)| \leq \epsilon$$

But then

$$\forall m > N_\epsilon : \sup_{x \in X} |f(x) - f_m(x)| \leq \epsilon$$

and thus, we have uniform convergence.

**Boundedness of  $f$ :**

Set  $\epsilon = \frac{1}{2}$  and choose  $N \in \mathbb{N}$  such that

$$\forall m, n \geq N : \sup_{x \in X} |f_n(x) - f_m(x)| < \frac{1}{2}$$

Then using the argument used for uniform convergence, we get

$$\begin{aligned} \forall x \in X : |f(x)| &\leq |f(x) - f_N(x)| + |f_N(x)| \leq \sup_{x \in X} |f(x) - f_N(x)| + \sup_{x \in X} |f_N(x)| \\ &\leq \frac{1}{2} + \sup_{x \in X} |f_N(x)| \end{aligned}$$

Thus,  $f$  is bounded.

**Continuity of  $f$ :**

Choose  $\epsilon > 0$  arbitrarily and let  $a \in X$  be an arbitrary point in our domain. Then by uniform convergence to  $f$  we have

$$\exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n > N_\epsilon : \sup_{x \in X} |f_n(x) - f(x)| < \frac{\epsilon}{3}$$

Since  $\forall n \in \mathbb{N}$   $f_n$  is continuous at  $a$  we have

$$\exists \delta > 0 \quad \text{such that} \quad |x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\epsilon}{3}$$

Combining these, we can obtain the following for sufficiently large values of  $n$ :

$$\forall x \in X : |x - a| < \delta \implies |f(x) - f(a)| \leq |f_n(x) - f(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < \epsilon$$

and thus  $f$  is continuous at  $a$ . Since  $a$  was chosen arbitrarily,  $f$  is continuous.  $\square$