Handout 6 - ECON703 (Fall 2023)

1 Lattices and Tarski Fixed Point Theorem

Definition 1.1 (Partial Order)**.** *A partial order $\preceq$ on a set X is a relation on $X \times X$ satisfying the following properties:*

1. **Reflexivity***:* $\forall x \in X: \; x \preceq x$
2. **Antisymmetry***:* $\forall x, y \in X: \; (x \preceq y \wedge y \preceq x) \implies (x = y)$
3. **Transitivity***:* $\forall x, y, z \in X: \; (x \preceq y \wedge y \preceq z) \implies (x \preceq z)$

A partially ordered set is an ordered pair $(X, \preceq)$ of a set X and a partial order $\preceq$ on X.

Definition 1.2 (Lattice)**.** *A partially ordered set $(X, \preceq)$ is called a lattice if*

$$\forall x, x' \in X : \exists! \overline{x} \in X \; s.t. \; \overline{x} = \sup\{x, x'\}$$
$$\forall x, x' \in X : \exists! \underline{x} \in X \; s.t. \; \underline{x} = \inf\{x, x'\}$$

Example 1.1 (Discrete Lattice)**.** *Consider the following partially ordered set $(X, \preceq)$, where*

$$X = \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$$

and

$$\forall x, y \in X : \quad (x \preceq y) \iff (y_1 \geq x_1 \wedge y_2 \geq x_2)$$

Then $(X, \preceq)$ is a lattice.

Definition 1.3 (Complete Lattice)**.** *A lattice $(X, \preceq)$ is called complete if every subset of $(X, \preceq)$ has a supremum and infimum.*

Theorem 1.1 (Tarski Fixed-Point Theorem / Knaster-Tarski Theorem)**.** *Let $(X, \preceq)$ be a complete lattice and $f : X \to X$ be a monotone function with respect to $\preceq$. Then, the set of fixed points of f on X forms a complete lattice under $\preceq$.*

Proof. See `https://eml.berkeley.edu/~fechenique/published/pftarski.pdf` for a proof of the theorem. □

2 Convex Sets

Definition 2.1 (Convex Set)**.** *A subset of Euclidean space $X \subset \mathbb{R}^n$ is called convex if*

$$\forall x, y \in X \; \forall \lambda \in [0,1] : \; \lambda x + (1 - \lambda) y \in X$$

There is no such thing as a concave set.

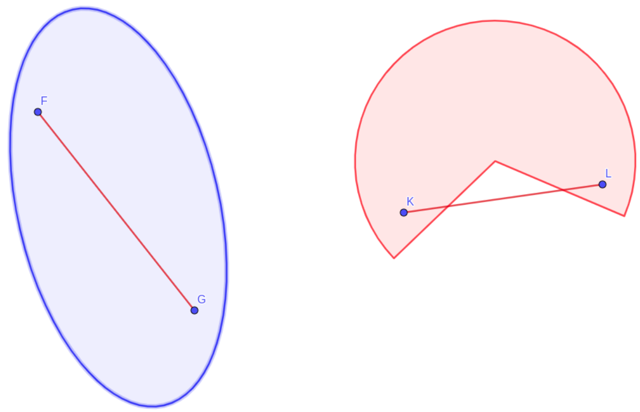

Figure 1: Convex set (left) and non-convex set (right)

3 (Quasi-)Convex Functions

Definition 3.1 (Convex / Concave Function)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function from* $\mathbb{R}^n$ *to* $\mathbb{R}$*.* f *is called concave if*

$$\forall x, y \in \mathbb{R}^n \; \forall \lambda \in [0, 1] : \; f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

It is called convex if

$$\forall x, y \in \mathbb{R}^n \; \forall \lambda \in [0, 1] : \; f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

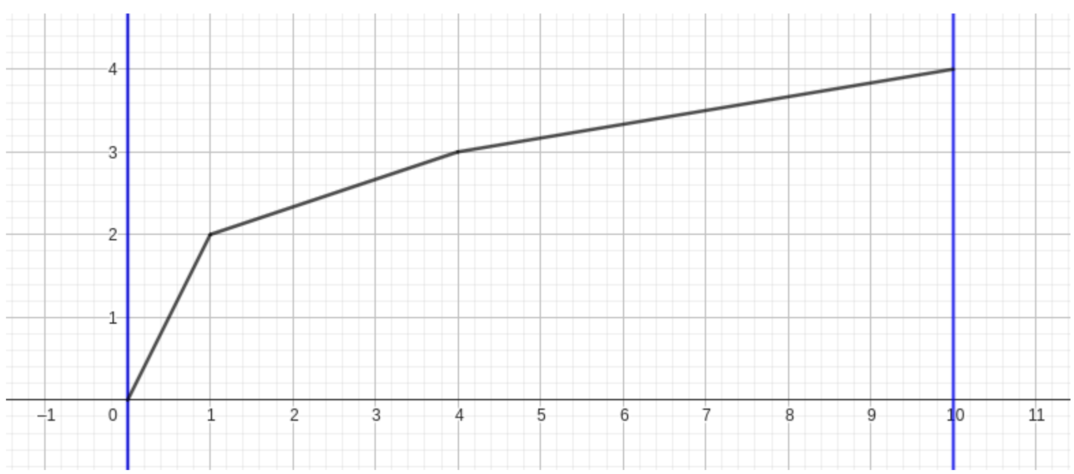

Figure 2: Example of Concave Function

Definition 3.2 (Strictly Convex / Strictly Concave Function)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function from* $\mathbb{R}^n$ *to* $\mathbb{R}$*.* f *is called strictly concave if*

$$\forall x, y \in \mathbb{R}^n \; \textit{with} \; x \neq y \; \forall \lambda \in (0, 1) : \; f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$$

It is called strictly convex if

$$\forall x, y \in \mathbb{R}^n \; \textit{with} \; x \neq y \; \forall \lambda \in (0, 1) : \; f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$$

Definition 3.3 (Quasiconvex / Quasiconcave Function)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function from* $\mathbb{R}^n$ *to* $\mathbb{R}$*.* f *is called quasi-concave if*

$$\forall x, y \in \mathbb{R}^n \; \forall \lambda \in [0, 1] : \; f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$$

It is called quasi-convex if

$$\forall x, y \in \mathbb{R}^n \; \forall \lambda \in [0,1] : \; f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$$

Definition 3.4 (Strictly Quasiconvex / Strictly Quasiconcave Function)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function from $\mathbb{R}^n$ to $\mathbb{R}$. f is called strictly quasi-concave if*

$$\forall x, y \in \mathbb{R}^n \; \textit{with} \, x \neq y \; \forall \lambda \in (0,1) : \; f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\}$$

It is called strictly quasi-convex if

$$\forall x, y \in \mathbb{R}^n \; \textit{with} \, x \neq y \; \forall \lambda \in (0,1) : \; f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}$$

Figure 3: Example of Strictly Concave Function

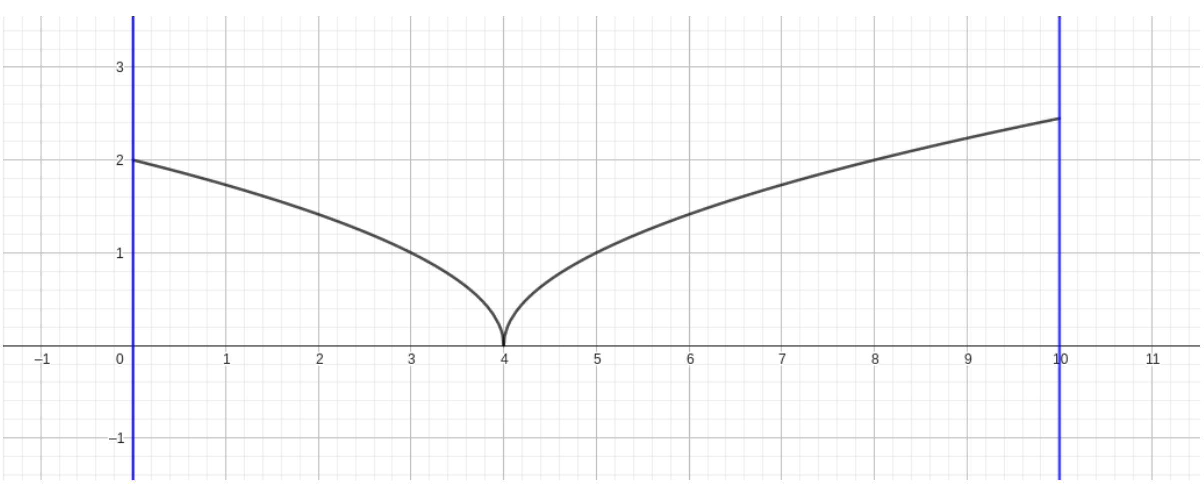

Figure 4: Example of a Strictly Quasiconvex Function