# Handout 7 - ECON703 (Fall 2023)

## 1 Graph of a Function

**Definition 1.1** (Graph of a Function)**.** *The graph of a function* $f : X \to Y$ *is defined as the set of ordered pairs* $\Gamma(f) = \{(x, y) \mid f(x) = y\} \subset X \times Y$.

**Definition 1.2** (Hypograph and Epigraph)**.** *The epigraph of a function* $f : X \to \mathbb{R}$ *is defined as*

$$Epi(f) = \{(x, y) \mid y \geq f(x)\} \subset X \times \mathbb{R}$$

*Its hypograph is defined as*

$$Hypo(f) = \{(x, y) \mid y \leq f(x)\} \subset X \times \mathbb{R}$$

We can extend those definitions to functions whose codomain is a partially ordered set.

## 2 Isoquants and Contour Sets

**Definition 2.1** (Isoquant)**.** *Let* $f : X \to \mathbb{R}$ *be a function from a set* $X$ *to* $\mathbb{R}$*. Then the isoquant of* $f$ *for the value* $z \in \mathbb{R}$ *is defined as*

$$\{x \in X \mid f(x) = z\}$$

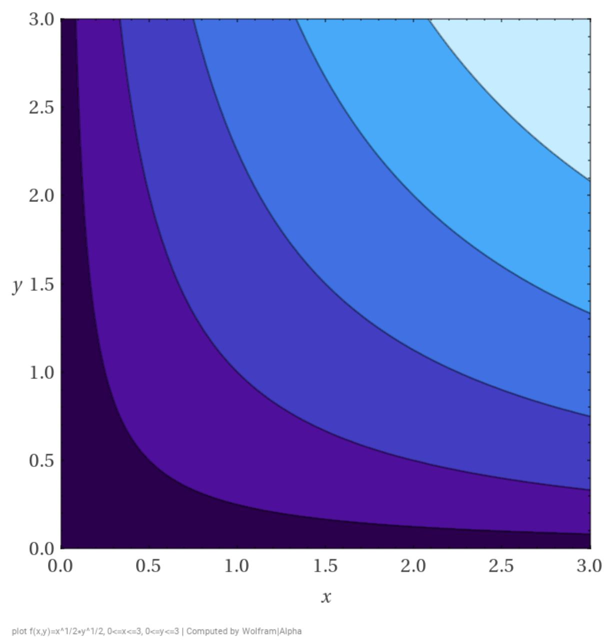


Figure 1: A contour plot of a Cobb-Douglas Function

**Definition 2.2** (Upper Contour Set)**.** *Let $f : X \to \mathbb{R}$ be a function from a set $X$ to $\mathbb{R}$. Then the upper contour set of $f$ at $z \in \mathbb{R}$ is defined as*

$$\mathcal{C}_z^+ = \{x \in X \mid f(x) \geq z\}$$

**Definition 2.3** (Lower Contour Set)**.** *Let $f : X \to \mathbb{R}$ be a function from a set $X$ to $\mathbb{R}$. Then the lower contour set of $f$ at $z \in \mathbb{R}$ is defined as*

$$\mathcal{C}_z^- = \{x \in X \mid f(x) \leq z\}$$

Like before, we can extend these definitions to functions whose codomain is a partially ordered set.

# 3 Differentiability

**Definition 3.1** (Directional Limits)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function from $\mathbb{R}$ to $\mathbb{R}$ and $I \subset \mathbb{R}$ be an interval. Let $a \in I$. We call $L \in \mathbb{R}$ the left limit of $f$ at $a$ if*

$$\lim_{x \to a^-} f(x) = L \iff \forall \epsilon > 0 \, \exists \delta > 0 \quad \textit{such that} \quad \forall x \in I : \, (0 < a - x < \delta) \implies (|f(x) - L| < \epsilon)$$

*We call $R \in \mathbb{R}$ the right limit of $f$ at $a$ if*

$$\lim_{x \to a^+} f(x) = R \iff \forall \epsilon > 0 \, \exists \delta > 0 \quad \textit{such that} \quad \forall x \in I : \, (0 < x - a < \delta) \implies (|f(x) - R| < \epsilon)$$

**Definition 3.2** (Differentiability in One Dimension)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function from $\mathbb{R}$ to $\mathbb{R}$. We say that $f$ is differentiable at $a \in \mathbb{R}$ if its derivative*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

*exists.*

**Definition 3.3** (Differentiable Function in One Dimension)**.** *Let $U \subset \mathbb{R}$ be a subset of the real line. We say that $f : U \to \mathbb{R}$ is a differentiable function if it is differentiable at all $x \in U$.*

**Example 3.1** (Differentiable Function)**.** *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then $f$ is a differentiable function.*

*Proof.*

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$$

Thus, the limit exists, and we even found the derivative. ☐

**Example 3.2** (Nondifferentiable Function)**.** *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} 2x & \textit{if} \quad x \leq 2 \\ 3x - 2 & \textit{if} \quad x > 2 \end{cases}$$

*Then $f$ is not differentiable at $x = 2$.*

*Proof.* We check sequences that approach 2 from the left and the right. In this case, these correspond to the left and right limits at $x = 2$.

Let $(h_n)_{n \in \mathbb{N}}$ be given by $h_n = 2^{-n}$.

$$\lim_{n \to \infty} \frac{f(2+h_n) - f(2)}{h_n} = \lim_{n \to \infty} \frac{3(2+h_n) - 2 - 4}{h_n} = \lim_{n \to \infty} \frac{3h_n}{h_n} = 3$$

$$\lim_{n \to \infty} \frac{f(2-h_n) - f(2)}{-h_n} = \lim_{n \to \infty} \frac{2(2-h_n) - 4}{-h_n} = \lim_{n \to \infty} \frac{-2h_n}{-h_n} = 2$$

Thus, we have shown that there are sequences converging to 2 such that the corresponding difference quotients converge to different values. Therefore, the limit, i.e., the derivative, does not exist at $x = 2$. ☐

# 4 Cardinal vs. Ordinal Properties

**Definition 4.1** (Ordinal Property)**.** *We say that a property of a function is ordinal if it is conserved under any strictly increasing transformation.*

**Example 4.1** (Quasiconcavity)**.** *Quasiconcavity is preserved under all strictly increasing transformations, as the upper and lower contour sets of a function do not change under strictly increasing transformations.*

**Definition 4.2** (Cardinal Property)**.** *A cardinal property of a function is a property that is not necessarily conserved under strictly increasing transformations.*

**Example 4.2** (Concavity)**.** *Consider the function* $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *given by* $f(x) = \sqrt{x}$*. Then* $f$ *is concave. Consider the transformation* $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *given by* $g(x) = x^4$*, which is strictly increasing. Then* $(g \circ f)(x) = x^2$ *is a strictly increasing transformation of* $f$ *that is not concave.*

# 5 Homogeneous and Homothetic Functions

**Definition 5.1** (Linear Cone)**.** *Let* $V \subset \mathbb{R}^n$ *be a subset of Euclidean space. We call* $V$ *a linear cone if*

$$\forall x \in V \, \forall \lambda \in \mathbb{R}_{\neq 0} : \quad \lambda x \in V$$

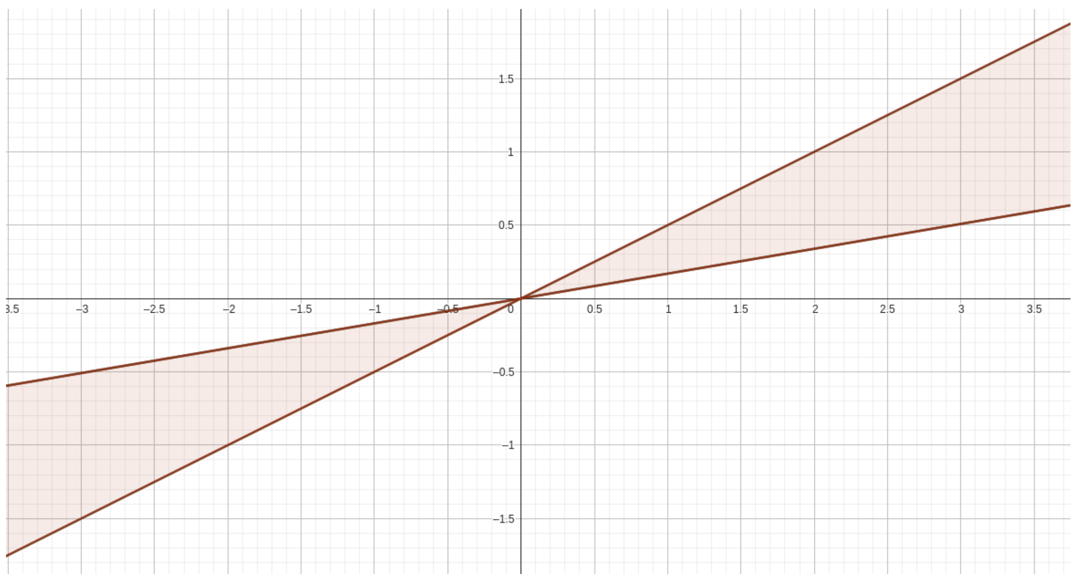


Figure 2: Example of a linear cone in $\mathbb{R}^2$

**Definition 5.2** (Homogeneous Function)**.** *Let* $V \subset \mathbb{R}^n$ *be a linear cone and* $f : V \to \mathbb{R}$ *be a function from* $V$ *to the real numbers. We say that* $f$ *is a homogeneous function of degree* $k \in \mathbb{N}$ *if*

$$\forall \lambda \in \mathbb{R}_{\neq 0} \, \forall x \in V : \quad f(\lambda x) = \lambda^k f(x)$$

**Definition 5.3** (Homothetic Function)**.** *We say a function* $f$ *is homothetic if it is a strictly increasing transformation of a homogeneous function.*

$\implies$ A homothetic function has "parallel" isoquants.