

Handout 2 - ECON703 (Fall 2023)

1 Supremum and Infimum

Example 1.1 (Approaching One). Let $X = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$. Then $\sup X = 1$.

Proof. Step 1: Show that 1 is an upper bound of X .

Observe that $\forall n \in \mathbb{N} : \frac{1}{n} > 0$. Thus $\forall n \in \mathbb{N} : 1 - \frac{1}{n} \leq 1$.

Step 2: Show that for any upper bound u of X , we have $u \geq 1$.

Let u be an upper bound of X and assume (for the sake of contradiction) that $u < 1$. But then:

$$\exists k \in \mathbb{N} \quad \text{s.t.} \quad \frac{1}{k} < 1 - u \implies \exists k \in \mathbb{N} \quad \text{s.t.} \quad 1 - \frac{1}{k} > u \implies \exists x \in X \quad \text{s.t.} \quad x > u$$

Contradiction! Thus, 1 is the supremum of X . □

Example 1.2 (Supremum and Infimum of a Square). Let $A = (0, 1) \times (0, 1) \subset \mathbb{R}^2$ and consider the following partial order $\preceq$ on $\mathbb{R}^2$:

$$\forall a, b \in A : (a \preceq b) \iff (b_1 \geq a_1 \wedge b_2 \geq a_2)$$

Then

$$\sup A = (1, 1) \quad \text{and} \quad \inf A = (0, 0)$$

Proof. We only consider the supremum; the infimum works analogously.

Step 1: Show that $(1, 1)$ is an upper bound of A with respect to the partial order.

$$\forall a \in A : a_1 < 1 \wedge a_2 < 1 \implies \forall a \in A : a \preceq (1, 1)$$

Therefore, $(1, 1)$ is an upper bound of A with respect to the partial order $\preceq$.

Step 2: Show that for any other upper bound u of A , we have $(1, 1) \preceq u$. Let u be an upper bound of A . Then:

$$\begin{aligned} \forall a \in A : a \preceq u &\implies \forall a \in A : u_1 \geq a_1 \wedge u_2 \geq a_2 \\ &\implies \forall x \in (0, 1) : u_1 \geq x \wedge u_2 \geq x \\ &\implies u_1 \geq 1 \wedge u_2 \geq 1 \\ &\implies (1, 1) \preceq u \end{aligned}$$

Thus, $(1, 1)$ is the supremum of A . □

2 Metric Spaces and Open/Closed Sets

Example 2.1 (Euclidean Space with Maximum Distance). Let $n \in \mathbb{N}$. Then $\mathbb{R}^n$ combined with the following operations is a vector space over the real numbers $\mathbb{R}$.

$$+_{\mathbb{V}} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \forall x, y \in \mathbb{R}^n \quad (x +_{\mathbb{V}} y)_i = x_i + y_i \quad \forall i = 1, \dots, n$$

$$\cdot_{\mathbb{V}} : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \forall x \in \mathbb{R}^n \forall \lambda \in \mathbb{R} \quad (\lambda \cdot_{\mathbb{V}} x)_i = \lambda x_i \quad \forall i = 1, \dots, n$$

We call it n -dimensional Euclidean space. The maximum distance on this space is

$$d(x, y) = \max_{i=1, \dots, n} |x_i - y_i|$$

Together, these form a metric space.

Proof. To show that this is a metric space, we have to show that d satisfies the properties of a metric.

1. $\forall v \in \mathbb{V} : d(v, v) = 0$
2. **Positivity:** $\forall u, v \in \mathbb{V}$ with $u \neq v : d(u, v) > 0$
3. **Symmetry:** $\forall u, v \in \mathbb{V} : d(u, v) = d(v, u)$
4. **Triangle Inequality:** $\forall u, v, w \in \mathbb{V} : d(u, w) \leq d(u, v) + d(v, w)$

We check them one by one.

1. Holds since $\forall v \in \mathbb{R}^n \forall i = 1, \dots, n : v_i - v_i = 0$ ✓
2. Holds since $u \neq v$ implies $\exists i \in \{1, \dots, n\}$ such that $|u_i - v_i| > 0$ ✓
3. Holds since $\forall a, b \in \mathbb{R} : |a - b| = |b - a|$ ✓
4. We check this one in a bit more detail. Let $x, y, z \in \mathbb{R}^n$

$$\begin{aligned} d(x, z) &= \max_{i=1, \dots, n} |x_i - z_i| = \max_{i=1, \dots, n} |x_i - y_i + y_i - z_i| \\ &\leq \max_{i=1, \dots, n} [|x_i - y_i| + |y_i - z_i|] \\ &\leq \max_{i=1, \dots, n} |x_i - y_i| + \max_{j=1, \dots, n} |y_j - z_j| = d(x, y) + d(y, z) \quad \checkmark \end{aligned}$$

□

One important thing to understand about open and closed sets is the following:

A subset is not like a door: it can be open, closed, both, or neither.

Example 2.2 (Open and Closed Sets). Consider the following sets as subsets of $\mathbb{R}^2$ and let $d(x, y)$ be the Euclidean distance between two points $x, y \in \mathbb{R}^2$.

- | | |
|--|---|
| 1. $\mathbb{R}^2$ | 6. $D = A \setminus \{x \in \mathbb{R}^2 \mid x_1 > 0\}$ |
| 2. $\emptyset$ | 7. $E = B \setminus \{x \in \mathbb{R}^2 \mid x_1 > 0\}$ |
| 3. $A = \{x \in \mathbb{R}^2 \mid d(x, (0, 0)) \leq 1\}$ | 8. $F = A \setminus \{x \in \mathbb{R}^2 \mid x_1 \geq 0\}$ |
| 4. $B = \{x \in \mathbb{R}^2 \mid d(x, (0, 0)) < 1\}$ | 9. $G = B \setminus \{x \in \mathbb{R}^2 \mid x_1 \geq 0\}$ |
| 5. $C = \{x \in \mathbb{R}^2 \mid d(x, (0, 0)) = 1\}$ | 10. $H = \{(0, 0)\}$ |

Are they open, closed, both, or neither?

3 Sequences and Series

Definition 3.1 (Convergent Sequence). Let (X, d_X) be a metric space and $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . We say that x_n converges to $\bar{x}$, denoted by $\lim_{n \rightarrow \infty} x_n = \bar{x}$ or $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$, if

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n > N_\epsilon : d_X(x_n, \bar{x}) < \epsilon$$

Example 3.1 (Geometric Series). Let $(x_n)_{n \in \mathbb{N}}$ be given by $x_n = \sum_{i=0}^{n-1} r^i$ for $r \in (0, 1)$. Then $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence with $\lim_{n \rightarrow \infty} x_n = \frac{1}{1-r}$.

Proof. First, we show that the following formula gives the value of a partial sum of the geometric series.

$$x_n = \sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r}$$

We will do so using a proof by induction.

- **Base Case:** $n = 1$

$$x_1 = \sum_{i=0}^0 r^i = r^0 = 1 = \frac{1 - r^1}{1 - r}$$

- **Induction Step:** Assume that $x_n = \frac{1 - r^n}{1 - r}$. Then

$$x_{n+1} = x_n + r^n = \frac{1 - r^n}{1 - r} + r^n = \frac{(1 - r^n) + (1 - r)r^n}{1 - r} = \frac{(1 - r^n) + (r^n - r^{n+1})}{1 - r} = \frac{1 - r^{n+1}}{1 - r}$$

Thus, we have shown that the assumed formula for the partial sums holds.

If the limit exists, this gives us the following statement.

$$\sum_{i=0}^{\infty} r^i = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} r^i = \lim_{n \rightarrow \infty} \frac{1 - r^n}{1 - r}$$

We can show that the second limit exists in the following way.

$$\lim_{n \rightarrow \infty} \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} \left(1 - \lim_{n \rightarrow \infty} r^n \right) = \frac{1}{1 - r} (1 - 0) = \frac{1}{1 - r}$$

where the second equality holds since $|r| < 1$. Thus, the series is convergent and converges to $\frac{1}{1 - r}$. □

Example 3.2 (Harmonic Series). Let $(x_n)_{n \in \mathbb{N}}$ be given by $x_n = \sum_{i=1}^n \frac{1}{i}$. Then $(x_n)_{n \in \mathbb{N}}$ diverges.

Proof. Observe that we can group the terms of the harmonic series as follows:

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \sum_{i=1}^{\infty} \left(\sum_{j=1}^{2^{i-1}} \frac{1}{2^{i-1} + j} \right) = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4} \right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \dots$$

But then

$$\forall i \in \mathbb{N} : \sum_{j=1}^{2^{i-1}} \frac{1}{2^{i-1} + j} \geq \frac{1}{2}$$

and thus

$$\sum_{i=1}^{\infty} \frac{1}{i} \geq \sum_{i=1}^{\infty} \frac{1}{2} = \infty$$

Thus $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges. □

4 Continuity

Definition 4.1 (Continuity). Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \rightarrow Y$ a function from X to Y . We say that f is continuous at a point $x \in X$ if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall x' \in X : d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon$$