

Handout 3 - ECON703 (Fall 2023)

1 Cauchy Sequences and Completeness

Definition 1.1 (Cauchy Sequence). Let (X, d_X) be a metric space and $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . Then $(x_n)_{n \in \mathbb{N}}$ is called a Cauchy sequence if

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \text{ such that } \forall m, n > N_\epsilon : d_X(x_n, x_m) < \epsilon$$

Example 1.1 (Partial Sums of the Geometric Series). The partial sums of the geometric series for $r = \frac{1}{2}$ are a Cauchy sequence.

Proof. Recall from Handout 2 that for $|r| < 1$

$$\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r} \quad \text{for } r = \frac{1}{2} \quad \sum_{i=0}^n 2^{-i} = 2(1 - 2^{-n-1}) = 2 - 2^{-n}$$

Then, we can find the following formula for the distance between two partial sums. Without loss of generality, let $m > n$.

$$0 < \left| \sum_{i=0}^m r^i - \sum_{i=0}^n r^i \right| = |2 - 2^{-m} - 2 + 2^{-n}| = |2^{-n} - 2^{-m}| = 2^{-n} - 2^{-m} < 2^{-n}$$

Choose some $\epsilon > 0$, then $\exists N_\epsilon \in \mathbb{N}$ such that $2^{-N_\epsilon} < \epsilon$. And $\forall n > N_\epsilon : 2^{-n} < \epsilon$. But then:

$$\forall m, n > N_\epsilon : \left| \sum_{i=0}^m r^i - \sum_{i=0}^n r^i \right| < \epsilon$$

□

Definition 1.2 (Complete Metric Space). Let (X, d_X) be a metric space. (X, d_X) is called complete if every Cauchy sequence in (X, d_X) converges in (X, d_X) , i.e. has a limit in X .

Theorem 1.1 (The Real Numbers). Consider the metric space $(\mathbb{R}, d)$ where $d(x, y) = |y - x|$. This metric space is complete.

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$. Then $\{x_n\}_{n \in \mathbb{N}}$ is bounded because

$$\exists N \in \mathbb{N} \forall m > N : d(x_{N+1}, x_m) < 1$$

$\Rightarrow \{x_n\}_{n > N}$ is bounded below by $x_{N+1} - 1$ and above by $x_{N+1} + 1$. Thus $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence and contains a convergent subsequence $(x_{n_i})_{i \in \mathbb{N}}$ (Bolzano-Weierstrass Theorem) with $\lim_{i \rightarrow \infty} x_{n_i} = x^*$. Choose an arbitrary $\epsilon > 0$.

$$\exists N_{\epsilon,1} \in \mathbb{N} \forall n, m > N_{\epsilon,1} : d(x_n, x_m) < \frac{\epsilon}{2} \quad \text{and} \quad \exists N_{\epsilon,2} \in \mathbb{N} \forall k > N_{\epsilon,2} : d(x_{n_k}, x^*) < \frac{\epsilon}{2}$$

Let $N_\epsilon = \max\{N_{\epsilon,1}, N_{\epsilon,2}\}$. Then

$$\forall k > N_\epsilon : d(x_k, x^*) \stackrel{\Delta\text{-ineq.}}{\leq} d(x_k, x_{n_k}) + d(x_{n_k}, x^*) \leq \epsilon$$

Thus $\lim_{n \rightarrow \infty} x_n = x^*$.

□

2 Bolzano-Weierstrass Theorem

Theorem 2.1 (Bolzano-Weierstrass in $\mathbb{R}$). Every infinite bounded sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ has a convergent subsequence.

Intuitive “Proof”. Call $n \in \mathbb{N}$ a peak of $(x_n)_{n \in \mathbb{N}}$ if $m > n \implies x_n > x_m$.

Consider the following cases:

1. $(x_n)_{n \in \mathbb{N}}$ has infinitely many peaks $n_1 < n_2 < n_3 < \dots$.
Then $(x_{n_j})_{j \in \mathbb{N}}$ is monotonically decreasing and bounded below. It is therefore convergent by the monotone convergence theorem.
2. $(x_n)_{n \in \mathbb{N}}$ has finitely many peaks. Let N be the last peak and let $n_1 = N + 1$.
 n_1 is not a peak $\implies \exists n_2 > n_1$ such that $x_{n_2} \geq x_{n_1}$.
 n_2 is not a peak $\implies \exists n_3 > n_2$ such that $x_{n_3} \geq x_{n_2}$ and so forth.
Then $(x_{n_j})_{j \in \mathbb{N}}$ is a bounded monotonically increasing sequence. It is convergent by the bounded convergence theorem.
3. $(x_n)_{n \in \mathbb{N}}$ has no peaks. Let $N = -1$. The argument provided in 2 applies.

□

Theorem 2.2 (Bolzano-Weierstrass). Every infinite bounded sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^n$ has a convergent subsequence.

Proof. We repeatedly apply Theorem 2.1 to the dimensions of $(x_n)_{n \in \mathbb{N}}$ to construct such a sequence.

- The first dimension of $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Apply Theorem 2.1 to the first dimension of $(x_n)_{n \in \mathbb{N}}$ to obtain a subsequence $(x_{n_i})_{i \in \mathbb{N}}$ whose first coordinate converges.
- The second dimension of $(x_{n_i})_{i \in \mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Apply Theorem 2.1 to the second dimension of $(x_{n_i})_{i \in \mathbb{N}}$ to obtain a sequence $(x_k)_{k \in \mathbb{N}}$ whose first and second coordinates converge. $(x_k)_{k \in \mathbb{N}}$ is a subsequence of $(x_{n_i})_{i \in \mathbb{N}}$ and thus a subsequence of $(x_n)_{n \in \mathbb{N}}$.
- ...

Iterate this process to obtain a subsequence of $(x_n)_{n \in \mathbb{N}}$ that converges in all of its n dimensions. This sequence is a convergent subsequence of $(x_n)_{n \in \mathbb{N}}$. □

3 Contraction Mappings

Definition 3.1 (Contraction Mapping). Let (X, d_X) be a metric space. A function $f : X \rightarrow X$ is called a contraction mapping on X if

$$\exists \beta \in [0, 1) \quad \text{such that} \quad \forall x, y \in X : \quad d_X(f(x), f(y)) \leq \beta d_X(x, y)$$

Example 3.1 (Example). A simple example of a contraction mapping is $f(x) = a + bx$ for $b \in [0, 1)$.

Example 3.2 (Counterexample). Let $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be given by $f(x) = x \left(1 - \frac{1}{1+x}\right)$.

Observe that this function shrinks distances between points. Without loss of generality, let $y > x$.

$$\begin{aligned} |f(y) - f(x)| &= y \left(1 - \frac{1}{1+y}\right) - x \left(1 - \frac{1}{1+x}\right) < y - x = |y - x| \\ \iff \frac{x}{1+x} < \frac{y}{1+y} &\iff x(1+y) < y(1+x) \iff x < y \quad \checkmark \end{aligned}$$

Next, let $y = x + 1$, and thus $|y - x| = 1$.

$$\begin{aligned} \lim_{x \rightarrow \infty} |f(y) - f(x)| &= \lim_{x \rightarrow \infty} \left| y \left(1 - \frac{1}{1+y}\right) - x \left(1 - \frac{1}{1+x}\right) \right| \\ &= \lim_{x \rightarrow \infty} (x+1) \left(1 - \frac{1}{2+x}\right) - x \left(1 - \frac{1}{1+x}\right) = \lim_{x \rightarrow \infty} \frac{1}{x+2} - \frac{1}{x+1} + 1 = 1 \end{aligned}$$

Even though f shrinks distances, its Lipschitz constant is not bounded away from one. Thus, it is not a contraction mapping.

Definition 3.2 (Fixed Point). Let $f : X \rightarrow X$ be a function from X to itself. We say $x \in X$ is a fixed point of f if $f(x) = x$.

Theorem 3.1 (Banach Fixed Point Theorem). Let (X, d_X) be a non-empty complete metric space. Let $f : X \rightarrow X$ be a contraction mapping on X . Then f has a unique fixed point x^* in X .

Proof. Choose $x_0 \in X$ arbitrarily. Define the sequence $(x_n)_{n \in \mathbb{N}}$ recursively by: $\forall n \in \mathbb{N}_0 : x_{n+1} = f(x_n)$. Note the following statement, which follows from iterated application of the contraction mapping definition.

$$d_X(x_{n+1}, x_n) \leq \beta^n d_X(x_1, x_0)$$

Next, we show that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ and, without loss of generality, let $m > n$

$$\begin{aligned} d_X(x_m, x_n) &\stackrel{\Delta\text{-ineq.}}{\leq} \sum_{i=n}^{m-1} d_X(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \beta^i d_X(x_0, x_1) = \beta^n d_X(x_0, x_1) \sum_{i=0}^{m-n-1} \beta^i \\ &\leq \beta^n d_X(x_0, x_1) \sum_{i=0}^{\infty} \beta^i = \beta^n d_X(x_0, x_1) \frac{1}{1-\beta} \end{aligned}$$

Choose an arbitrary $\epsilon > 0$. Then

$$\exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \beta^{N_\epsilon} \frac{d_X(x_0, x_1)}{1-\beta} < \epsilon$$

and thus

$$\forall m, n > N_\epsilon : d_X(x_m, x_n) < \epsilon$$

Therefore, $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Since (X, d_X) is complete, $(x_n)_{n \in \mathbb{N}}$ converges in (X, d_X) .

Let $x^* = \lim_{n \rightarrow \infty} x_n$.

$$d_X(x^*, f(x^*)) \leq d_X(x^*, x_m) + d_X(x_m, f(x^*)) \leq d_X(x^*, x_m) + \beta d_X(x_{m-1}, x^*)$$

But

$$\lim_{m \rightarrow \infty} d_X(x^*, x_m) = 0$$

Thus, we can bound $d_X(x^*, f(x^*))$ by arbitrarily small positive values. Therefore, $d_X(x^*, f(x^*)) = 0$, meaning that x^* is a fixed point of f .

Assume (for the sake of contradiction) that there is a second fixed point; call it x^{**} .

$$d_X(x^*, x^{**}) = d_X(f(x^*), f(x^{**})) \leq \beta d_X(x^*, x^{**})$$

Since $\beta < 1$, this implies $d_X(x^*, x^{**}) = 0$ and therefore $x^* = x^{**}$. Thus x^* must be the unique fixed point of f . $\square$

Example 3.3 (Dynamic Programming). **Do not worry!**

You will see this in much more detail during your macroeconomics class! So I will leave out all the nuance here. Let $C_b(\mathbb{R})$ be the space of continuous and bounded functions on the real numbers. Let $T : C_b(\mathbb{R}) \rightarrow C_b(\mathbb{R})$ be defined pointwise by:

$$T[v](x) = \max_{z \in \Gamma(x)} (u(x - z) + \beta v(z))$$

where Γ is a feasibility correspondence.

Then we can show (under some conditions) that T is a contraction mapping on $C_b(\mathbb{R})$. It thus has a fixed point, which will be useful in **many** economic problems. (Dynamic Optimization)

4 Convergence of Functions

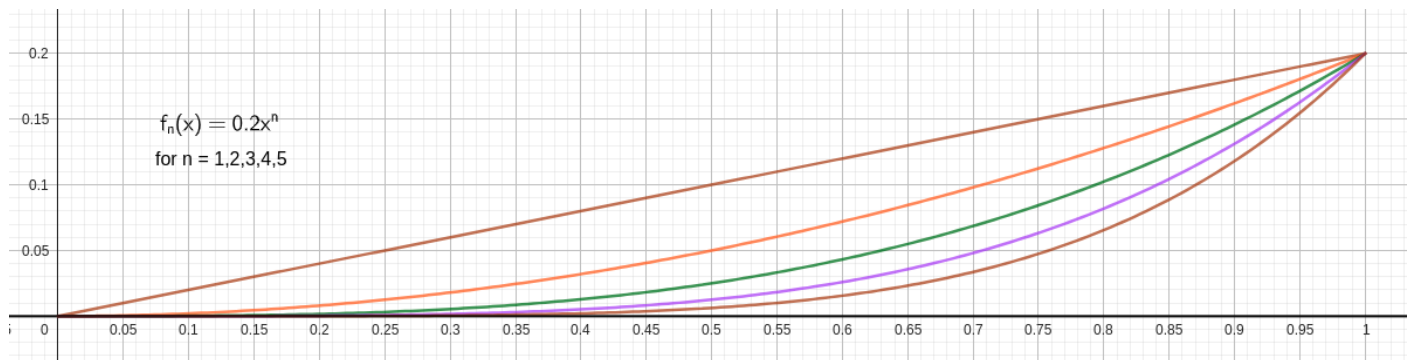
Definition 4.1 (Pointwise Convergence). Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$. We say that $(f_n)_{n \in \mathbb{N}}$ converges pointwise to some function $f : \mathbb{R} \rightarrow \mathbb{R}$ if

$$\forall x \in \mathbb{R} : \lim_{n \rightarrow \infty} f_n(x) = f(x)$$

Definition 4.2 (Uniform Convergence). Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$. We say that $(f_n)_{n \in \mathbb{N}}$ converges uniformly to some function $f : \mathbb{R} \rightarrow \mathbb{R}$ if

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = 0$$

Example 4.1 (Pointwise but not Uniform Convergence). Let $f_n : [0, 1) \rightarrow [0, 1)$ be given by $f_n(x) = x^n$. Then f_n converges pointwise to $f(x) = 0$. However, f_n does not converge uniformly.



Proof. Take $x \in [0, 1)$ arbitrarily. Then $\lim_{n \rightarrow \infty} x^n = 0$. Thus f_n converges pointwise to $f(x) = 0$.

f is the only candidate for uniform convergence. However,

$$\forall n \in \mathbb{N} : \sup_{x \in [0, 1)} |x^n - 0| = 1$$

Thus, f_n does not converge uniformly. □