

Handout 4 - ECON703 (Fall 2023)

1 Cluster Points and the Squeeze Theorem

Definition 1.1 (Cluster Point / Accumulation Point). Let (X, d_X) be a metric space and $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . We say that $y \in X$ is a cluster point of $(x_n)_{n \in \mathbb{N}}$ if $(x_n)_{n \in \mathbb{N}}$ has a subsequence converging to y .

Theorem 1.1 (Supremum of the Set of Cluster Points). Let $(x_n)_{n \in \mathbb{N}}$ be a bounded^a sequence in $\mathbb{R}$ and $\mathbb{Y} \subset \mathbb{R}$ be the set of cluster points of $(x_n)_{n \in \mathbb{N}}$. Then $\sup \mathbb{Y} = \limsup_{n \rightarrow \infty} x_n$.

^aWe can make this work with unbounded sequences, but then $\limsup_{n \rightarrow \infty} x_n = \infty$ becomes a possibility.

Proof. Recall the definition of the limit superior:

$$b_n = \sup\{x_k \mid k \geq n\} \quad \text{i.e.} \quad \lim_{n \rightarrow \infty} b_n = \limsup_{n \rightarrow \infty} x_n$$

Then, as an exercise, one can show that

$$\exists (x_{n_i})_{i \in \mathbb{N}} \quad \text{such that} \quad \lim_{i \rightarrow \infty} x_{n_i} = \limsup_{n \rightarrow \infty} x_n$$

and thus

$$\limsup_{n \rightarrow \infty} x_n \in \mathbb{Y} \quad \text{which gives us} \quad \sup \mathbb{Y} \geq \limsup_{n \rightarrow \infty} x_n$$

Let $(x_{n_k})_{k \in \mathbb{N}}$ be a convergent subsequence of $(x_n)_{n \in \mathbb{N}}$. Then we have $\lim_{k \rightarrow \infty} x_{n_k} = x^* \in \mathbb{Y}$. Then

$$\forall k \in \mathbb{N} : x_{n_k} \leq b_{n_k}$$

Passing to the limit gives:

$$x^* = \lim_{k \rightarrow \infty} x_{n_k} \leq \lim_{k \rightarrow \infty} b_{n_k} = \limsup_{n \rightarrow \infty} x_n$$

and thus $\forall y \in \mathbb{Y} : y \leq \limsup_{n \rightarrow \infty} x_n$. □

Analogously, this applies to the limit inferior of this sequence as a lower bound of $\mathbb{Y}$.

Theorem 1.2 (Squeeze Theorem / Sandwich Theorem). Let $f : [a, b] \rightarrow \mathbb{R}$, $g : [a, b] \rightarrow \mathbb{R}$, and $h : [a, b] \rightarrow \mathbb{R}$ and $c \in [a, b]$. We can also make this work if f , g , and h are only defined on $[a, b] \setminus \{c\}$. Assume that

$$\forall x \in [a, b] \setminus \{c\} : g(x) \leq f(x) \leq h(x)$$

and

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$$

Then $\lim_{x \rightarrow c} f(x) = L$.

Proof.

$$\begin{aligned} L &= \lim_{x \rightarrow c} g(x) \leq \liminf_{x \rightarrow c} f(x) \leq \limsup_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} h(x) = L \\ \implies \liminf_{x \rightarrow c} f(x) &= \limsup_{x \rightarrow c} f(x) = L \\ \implies \lim_{x \rightarrow c} f(x) &= L \end{aligned}$$

The second implication follows from Homework Problem 14. □

Example 1.1 (Strange Sine Function). Consider the function $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ with $f(x) = x^2 \sin(\frac{1}{x})$. Then since $\sin(x) \in [-1, 1]$, we have:

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Then we can find the following limits:

$$\lim_{x \rightarrow 0} -x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x^2 = 0$$

so the squeeze theorem implies that $\lim_{x \rightarrow 0} f(x) = 0$.

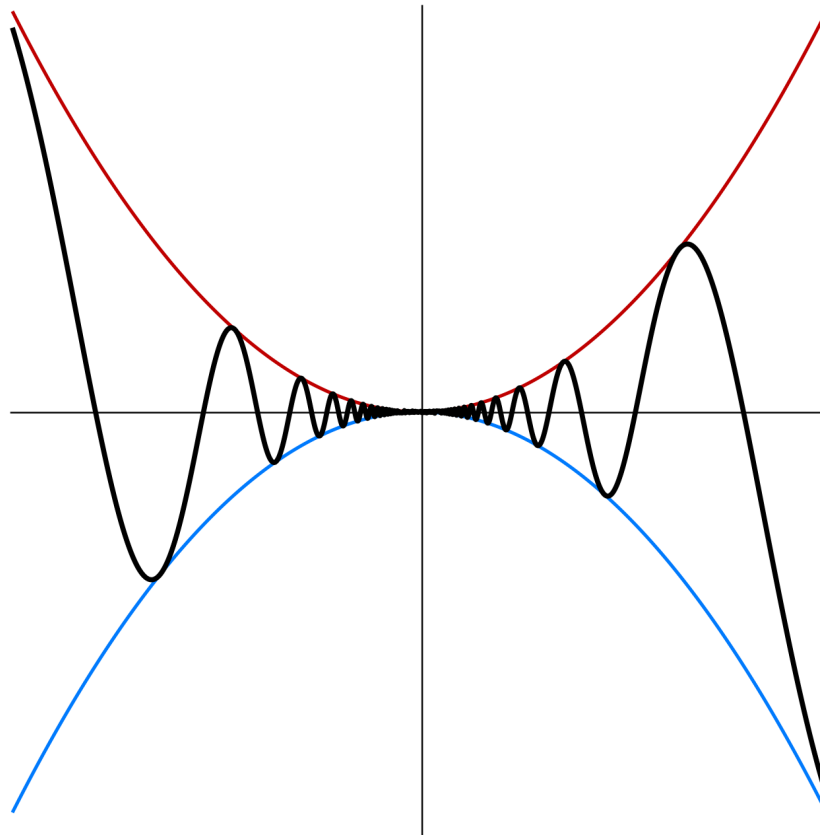


Figure 1: Source: https://en.wikipedia.org/wiki/File:Inst_satsen.png

2 Lipschitz-Continuity

Definition 2.1 (Lipschitz-Continuity). Let (X, d_X) and (Y, d_Y) be two metric spaces. We say a function $f : X \rightarrow Y$ is Lipschitz continuous if

$$\exists K \in \mathbb{R}_{\geq 0} \quad \text{such that} \quad \forall x, x' \in X : d_Y(f(x), f(x')) \leq K d_X(x, x')$$

We call any such K a Lipschitz constant of f and the smallest such K the dilation of f .

Example 2.1 (Absolute Value). The function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = |x|$ is Lipschitz continuous with dilation 1.

Theorem 2.1 (Lipschitz-Continuous Functions are Continuous). Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $f : X \rightarrow Y$ be a Lipschitz continuous function with dilation $K \in \mathbb{R}$. Then f is continuous.

Proof. Since f is Lipschitz continuous, we know that

$$\forall x, x' \in X : d_Y(f(x), f(x')) \leq K d_X(x, x')$$

If $K = 0$, then $d_Y(f(x), f(x')) = 0$ for all $x, x' \in X$ and continuity is immediate. Now consider the case $K > 0$. Choose an arbitrary $\epsilon > 0$ and set $\delta = \frac{\epsilon}{K}$. Then

$$d_X(x, x') < \delta \implies d_X(x, x') < \frac{\epsilon}{K} \implies K d_X(x, x') < \epsilon \implies d_Y(f(x), f(x')) < \epsilon$$

Thus, f is continuous. □

3 Continuity

Theorem 3.1 (Sequence Definition of Continuity). Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \rightarrow Y$. f is continuous at $x \in X$ if and only if

$$\forall (x_n)_{n \in \mathbb{N}} \text{ with } \lim_{n \rightarrow \infty} x_n = x : \lim_{n \rightarrow \infty} f(x_n) = f(x)$$

*Proof*¹. **Step 1:** If this property holds, f is continuous at x .

We show the contrapositive, i.e., if the epsilon-delta definition does not hold at x , then the above sequence property is violated. Thus, assume that f is not continuous at x , i.e.

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0 \exists x' \in X \text{ with } d_X(x', x) < \delta \wedge d_Y(f(x'), f(x)) \geq \epsilon$$

Let ϵ_0 be the $\epsilon > 0$ at which our continuity definition is violated. Now consider the following sequences

$$(\delta_i)_{i \in \mathbb{N}} \text{ with } \delta_i = 2^{-i}$$

and

$$(x_i)_{i \in \mathbb{N}} \text{ with } d_X(x, x_i) < \delta_i \wedge d_Y(f(x_i), f(x)) \geq \epsilon_0$$

But then

$$\lim_{i \rightarrow \infty} x_i = x \text{ and } \lim_{i \rightarrow \infty} f(x_i) \neq f(x)$$

Thus, the property under investigation is violated.

Step 2: If f is continuous at x , this property holds.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence with $\lim_{n \rightarrow \infty} x_n = x$ and choose an arbitrary $\epsilon > 0$. Since f is continuous at x , we have

$$\exists \delta > 0 \text{ such that } \forall x' \in X \text{ with } d_X(x, x') < \delta : d_Y(f(x), f(x')) < \epsilon$$

But since $\lim_{n \rightarrow \infty} x_n = x$, we have

$$\exists N_\delta \in \mathbb{N} \text{ such that } \forall n > N_\delta : d_X(x_n, x) < \delta$$

and thus

$$\forall n > N_\delta : d_Y(f(x_n), f(x)) < \epsilon$$

But then

$$\lim_{n \rightarrow \infty} f(x_n) = f(x)$$

□