

Handout 5 - ECON703 (Fall 2023)

1 Vector Spaces and Norms

In the following, whenever we use the term field ($\mathbb{F}$), you can think of $\mathbb{R}$ together with our usual addition and multiplication. We can define many of these objects more generally, but let us stay in the reals for now.

Definition 1.1 (Vector Space). A vector space $(\mathbb{V}, +_{\mathbb{V}}, \cdot_{\mathbb{V}})$ over a field $\mathbb{F}$ (with addition $+_{\mathbb{F}}$ and multiplication $\cdot_{\mathbb{F}}$) is a non-empty set $\mathbb{V}$ together with two binary operations $+_{\mathbb{V}} : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$ and $\cdot_{\mathbb{V}} : \mathbb{F} \times \mathbb{V} \rightarrow \mathbb{V}$ such that the following properties hold.

- Associativity of Vector Addition:

$$\forall u, v, w \in \mathbb{V} : u + (v + w) = (u + v) + w$$

- Commutativity of Vector Addition:

$$\forall u, v \in \mathbb{V} : u + v = v + u$$

- Existence of an Identity Element of Vector Addition:

$$\exists 0_{\mathbb{V}} \in \mathbb{V} \text{ s.t. } \forall v \in \mathbb{V} : v + 0_{\mathbb{V}} = v$$

- Existence of Inverse Elements of Vector Addition:

$$\forall v \in \mathbb{V} \exists (-v) \in \mathbb{V} \text{ s.t. } v + (-v) = 0_{\mathbb{V}}$$

- Compatibility of Scalar Multiplication with Field Multiplication:

$$\forall v \in \mathbb{V} \forall \lambda, \mu \in \mathbb{F} : \lambda \cdot_{\mathbb{V}} (\mu \cdot_{\mathbb{V}} v) = (\lambda \cdot_{\mathbb{F}} \mu) \cdot_{\mathbb{V}} v$$

- Existence of an Identity Element of Scalar Multiplication:

$$\forall v \in \mathbb{V} : 1_{\mathbb{F}} \cdot_{\mathbb{V}} v = v$$

- Distributivity of Scalar Multiplication w.r.t. Vector Addition:

$$\forall u, v \in \mathbb{V} \forall \lambda \in \mathbb{F} : \lambda \cdot_{\mathbb{V}} (u +_{\mathbb{V}} v) = \lambda \cdot_{\mathbb{V}} v +_{\mathbb{V}} \lambda \cdot_{\mathbb{V}} u$$

- Distributivity of Scalar Multiplication w.r.t. Field Addition:

$$\forall v \in \mathbb{V} \forall \lambda, \mu \in \mathbb{F} : (\lambda +_{\mathbb{F}} \mu) \cdot_{\mathbb{V}} v = \lambda \cdot_{\mathbb{V}} v +_{\mathbb{V}} \mu \cdot_{\mathbb{V}} v$$

Example 1.1 (Two-Dimensional Real Vectors). The set of all ordered pairs of real numbers, i.e., $\mathbb{R}^2$ together with component-wise addition and scalar multiplication, is a vector space over $\mathbb{R}$.

$$+ : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{defined by} \quad (x, y) + (a, b) = (x + a, y + b)$$

$$\cdot : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{defined by} \quad \lambda \cdot (x, y) = (\lambda x, \lambda y)$$

Proof. Check the properties...

□

Definition 1.2 (Norm). A function $\|\cdot\| : \mathbb{V} \rightarrow \mathbb{R}$ from a vector space $\mathbb{V}$ to the underlying field $\mathbb{R}$ is called a norm if it satisfies the following properties.

1. **Triangle Inequality:**

$$\forall u, v \in \mathbb{V} : \quad \|u + v\| \leq \|u\| + \|v\|$$

2. **Absolute Homogeneity:**

$$\forall v \in \mathbb{V}, \lambda \in \mathbb{F} : \quad \|\lambda v\| = |\lambda| \|v\|$$

3. **Positive Definiteness:**

$$\forall v \in \mathbb{V} : \quad \|v\| \geq 0 \quad \text{and} \quad \|v\| = 0_{\mathbb{F}} \iff v = 0_{\mathbb{V}}$$

A function that satisfies properties 1 and 2 is called a seminorm.

Example 1.2 (ℓ_p -Norm). For $p \in [1, \infty)$, the following formula defines the ℓ_p -norm.

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

This is a norm on an n -dimensional Euclidean space.

Theorem 1.1 (Distance induced by a Norm). A norm $\|\cdot\| : \mathbb{V} \rightarrow \mathbb{R}$ on a vector space $(\mathbb{V}, +_{\mathbb{V}}, \cdot_{\mathbb{V}})$ induces a distance on $(\mathbb{V}, +_{\mathbb{V}}, \cdot_{\mathbb{V}})$ as follows:

$$d(x, y) = \|x - y\|$$

Proof. Check the properties of a distance:

- $\forall v \in \mathbb{V} : d(v, v) = 0$ Holds by positive definiteness of the underlying norm ✓
- **Positivity:** Holds by positivity of the underlying norm ✓
- **Symmetry:** Holds by absolute homogeneity with $\lambda = -1$ ✓
- **Triangle Inequality:** Let $x, y, z \in \mathbb{V}$

$$d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y) \quad \checkmark$$

□

2 Inner Products and the Cauchy-Schwarz Inequality

Definition 2.1 (Inner Product). An inner product on a vector space $\mathbb{V}$ over $\mathbb{R}$ is a function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$ satisfying the following properties.

- **Symmetry:** $\forall v, u \in \mathbb{V} : \langle v, u \rangle = \langle u, v \rangle$
- **Linearity in the First Argument:** $\forall u, v, w \in \mathbb{V} \forall \lambda, \mu \in \mathbb{F} : \langle \lambda v + \mu u, w \rangle = \lambda \langle v, w \rangle + \mu \langle u, w \rangle$
- **Positive Definiteness:** $\forall v \in \mathbb{V} : (v \neq 0) \implies (\langle v, v \rangle > 0)$

Example 2.1 (Canonical Inner Product on Euclidean Space). The canonical inner product on $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, is given by

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i$$

Proof. Check the properties of the inner product definition: symmetry, linearity in the first argument, and positive definiteness all hold. □

Definition 2.2 (Inner Product Space). An inner product space is a vector space $\mathbb{V}$ over some field $\mathbb{F}$ together with an inner product $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{F}$.

Theorem 2.1 (Cauchy-Schwarz Inequality). Let $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ be an inner product space. Then

$$\forall u, v \in \mathbb{V} : |\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle$$

In Euclidean space with the standard inner product, the Cauchy-Schwarz inequality becomes:

$$\left(\sum_{i=1}^n u_i v_i \right)^2 \leq \left(\sum_{i=1}^n u_i^2 \right) \left(\sum_{i=1}^n v_i^2 \right)$$

with equality if and only if u and v are linearly dependent.

Proof. Due to time constraints: see https://en.wikipedia.org/wiki/Cauchy-Schwarz_inequality#Proofs □

3 The Space of Continuous and Bounded Functions

Definition 3.1 (Uniformly Bounded Set of Functions). Let X be an arbitrary set and $\mathcal{F} = \{f_i : X \rightarrow \mathbb{R} \mid i \in \mathcal{I}\}$ be a set of functions indexed by some index set $\mathcal{I}$. We say that $\mathcal{F}$ is bounded uniformly if

$$\exists M \in \mathbb{R} \quad \text{such that} \quad \forall i \in \mathcal{I} \forall x \in X : |f_i(x)| \leq M$$

Definition 3.2 (Space of Continuous and Bounded Functions). The space of continuous and bounded functions from a set $X \subset \mathbb{R}^n$ to $\mathbb{R}$, denoted by $\mathcal{C}_b(X)$, is defined as the vector space consisting of the set

$$\{f : X \rightarrow \mathbb{R} \mid f \text{ continuous and bounded}\}$$

together with pointwise addition and scalar multiplication.

$$\begin{aligned} + : \mathcal{C}_b(X) \times \mathcal{C}_b(X) &\rightarrow \mathcal{C}_b(X) & \text{given by} & & \forall f, g \in \mathcal{C}_b(X) \forall x \in X : (f+g)(x) &= f(x) + g(x) \\ \cdot : \mathbb{R} \times \mathcal{C}_b(X) &\rightarrow \mathcal{C}_b(X) & \text{given by} & & \forall f \in \mathcal{C}_b(X) \forall \lambda \in \mathbb{R} \forall x \in X : (\lambda f)(x) &= \lambda f(x) \end{aligned}$$

One important point is that not all of these functions have to be bounded by the same bound M . In fact, if we assume that they are, this is not a vector space.

Theorem 3.1 (The space of Continuous and Bounded Functions with the sup-norm is a Complete Metric Space). $\mathcal{C}_b(X)$ together with the operations defined above and the distance induced by the sup-norm

$$d(f, g) = \sup_{x \in X} |g(x) - f(x)|$$

is a complete metric space.

Proof. Let $f_i : X \rightarrow \mathbb{R}$ be a Cauchy sequence with respect to the sup-distance, i.e.

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n, m > N_\epsilon : \sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon$$

Then $\forall x \in X$ the sequence $(f_i(x))_{i \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and thus convergent. This means that the pointwise limit of our function sequence exists. Let $f : X \rightarrow \mathbb{R}$ be the pointwise limit of f_i , i.e.

$$\forall x \in X : f(x) = \lim_{i \rightarrow \infty} f_i(x)$$

This will be our candidate for the limit of our function sequence in the sup-distance.

Uniform Convergence:

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n, m > N_\epsilon : \sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon$$

and thus, by continuity of the absolute value, we have

$$\forall x \in X : |f(x) - f_m(x)| = \left| \lim_{i \rightarrow \infty} f_i(x) - f_m(x) \right| = \lim_{i \rightarrow \infty} |f_i(x) - f_m(x)| \leq \epsilon$$

But then

$$\forall m > N_\epsilon : \sup_{x \in X} |f(x) - f_m(x)| \leq \epsilon$$

and thus, we have uniform convergence.

Boundedness of f :

Set $\epsilon = \frac{1}{2}$ and choose $N \in \mathbb{N}$ such that

$$\forall m, n \geq N : \sup_{x \in X} |f_n(x) - f_m(x)| < \frac{1}{2}$$

Then, using the argument used for uniform convergence, we get

$$\begin{aligned} \forall x \in X : |f(x)| &\leq |f(x) - f_N(x)| + |f_N(x)| \leq \sup_{x \in X} |f(x) - f_N(x)| + \sup_{x \in X} |f_N(x)| \\ &\leq \frac{1}{2} + \sup_{x \in X} |f_N(x)| \end{aligned}$$

Thus, f is bounded.

Continuity of f :

Choose an arbitrary $\epsilon > 0$ and an arbitrary point $a \in X$ in our domain. Then by uniform convergence to f we have

$$\exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n > N_\epsilon : \sup_{x \in X} |f_n(x) - f(x)| < \frac{\epsilon}{3}$$

Choose some $n > N_\epsilon$. Since f_n is continuous at a , we have

$$\exists \delta > 0 \quad \text{such that} \quad \|x - a\| < \delta \implies |f_n(x) - f_n(a)| < \frac{\epsilon}{3}$$

Combining these, we can obtain the following for sufficiently large values of n :

$$\forall x \in X : \|x - a\| < \delta \implies |f(x) - f(a)| \leq |f_n(x) - f(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < \epsilon$$

and thus f is continuous at a . Since a was chosen arbitrarily, f is continuous. □