

Handout 6 - ECON703 (Fall 2023)

1 Lattices and Tarski Fixed Point Theorem

Definition 1.1 (Partial Order). A partial order $\preceq$ on a set X is a relation on $X \times X$ satisfying the following properties:

1. **Reflexivity:** $\forall x \in X : x \preceq x$
2. **Antisymmetry:** $\forall x, y \in X : (x \preceq y \wedge y \preceq x) \implies (x = y)$
3. **Transitivity:** $\forall x, y, z \in X : (x \preceq y \wedge y \preceq z) \implies (x \preceq z)$

A partially ordered set is an ordered pair $(X, \preceq)$ of a set X and a partial order $\preceq$ on X .

Definition 1.2 (Lattice). A partially ordered set $(X, \preceq)$ is called a lattice if

$$\begin{aligned} \forall x, x' \in X : \exists! \bar{x} \in X \text{ s.t. } \bar{x} &= \sup\{x, x'\} \\ \forall x, x' \in X : \exists! \underline{x} \in X \text{ s.t. } \underline{x} &= \inf\{x, x'\} \end{aligned}$$

Example 1.1 (Discrete Lattice). Consider the following partially ordered set $(X, \preceq)$, where

$$X = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset \mathbb{R}^2$$

and

$$\forall x, y \in X : (x \preceq y) \iff (y_1 \geq x_1 \wedge y_2 \geq x_2)$$

Then $(X, \preceq)$ is a lattice.

Definition 1.3 (Complete Lattice). A lattice $(X, \preceq)$ is called complete if every subset of $(X, \preceq)$ has a supremum and an infimum.

Theorem 1.1 (Tarski Fixed-Point Theorem / Knaster-Tarski Theorem). Let $(X, \preceq)$ be a complete lattice and $f : X \rightarrow X$ be a monotone function with respect to $\preceq$. Then, the set of fixed points of f on X forms a complete lattice under $\preceq$.

Proof. See <https://eml.berkeley.edu/~fechenique/published/pftarski.pdf> for a proof of the theorem. □

2 Convex Sets

Definition 2.1 (Convex Set). A subset of Euclidean space $X \subset \mathbb{R}^n$ is called convex if

$$\forall x, y \in X \forall \lambda \in [0, 1] : \lambda x + (1 - \lambda)y \in X$$

There is no such thing as a concave set.

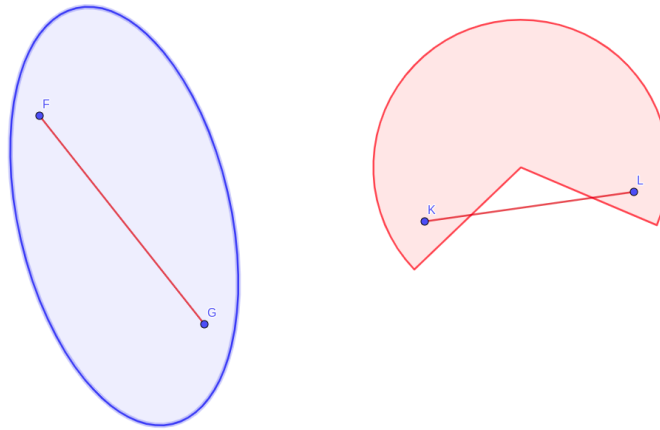


Figure 1: Convex set (left) and non-convex set (right)

3 (Quasi-)Convex Functions

Definition 3.1 (Convex / Concave Function). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function from $\mathbb{R}^n$ to $\mathbb{R}$. f is called concave if

$$\forall x, y \in \mathbb{R}^n \forall \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

It is called convex if

$$\forall x, y \in \mathbb{R}^n \forall \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

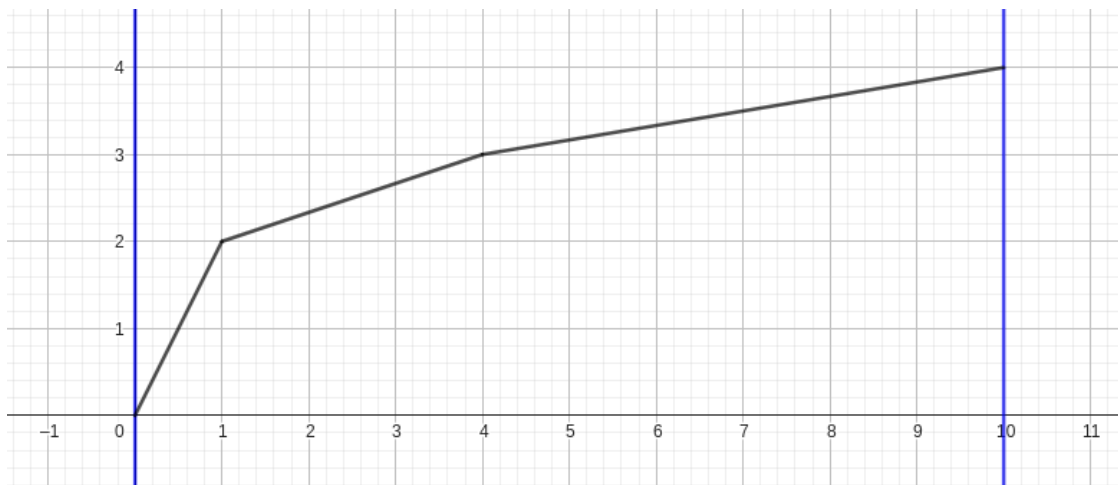


Figure 2: Example of Concave Function

Definition 3.2 (Strictly Convex / Strictly Concave Function). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function from $\mathbb{R}^n$ to $\mathbb{R}$. f is called strictly concave if

$$\forall x, y \in \mathbb{R}^n \text{ with } x \neq y \forall \lambda \in (0, 1) : f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$$

It is called strictly convex if

$$\forall x, y \in \mathbb{R}^n \text{ with } x \neq y \forall \lambda \in (0, 1) : f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$$

Definition 3.3 (Quasiconvex / Quasiconcave Function). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function from $\mathbb{R}^n$ to $\mathbb{R}$. f is called quasi-concave if

$$\forall x, y \in \mathbb{R}^n \forall \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$$

It is called quasi-convex if

$$\forall x, y \in \mathbb{R}^n \forall \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$$

Definition 3.4 (Strictly Quasiconvex / Strictly Quasiconcave Function). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function from $\mathbb{R}^n$ to $\mathbb{R}$. f is called strictly quasi-concave if

$$\forall x, y \in \mathbb{R}^n \text{ with } x \neq y \forall \lambda \in (0, 1) : f(\lambda x + (1 - \lambda)y) > \min\{f(x), f(y)\}$$

It is called strictly quasi-convex if

$$\forall x, y \in \mathbb{R}^n \text{ with } x \neq y \forall \lambda \in (0, 1) : f(\lambda x + (1 - \lambda)y) < \max\{f(x), f(y)\}$$

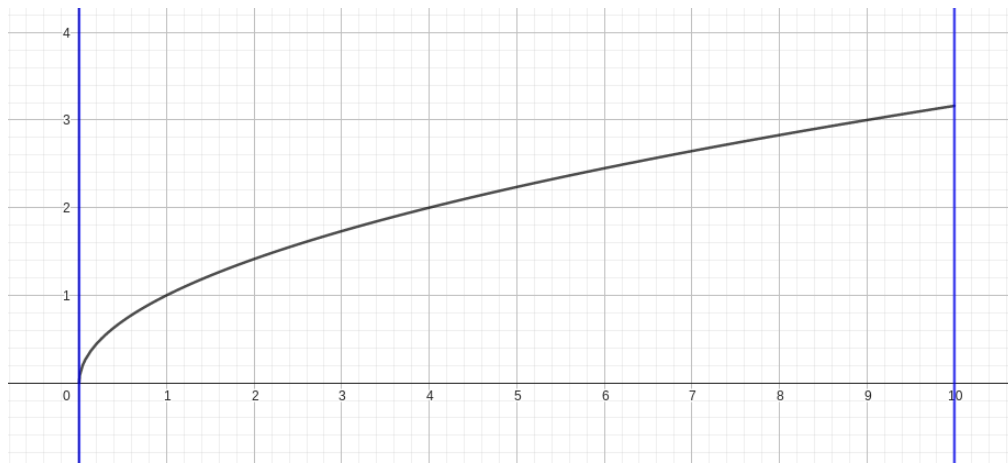


Figure 3: Example of a strictly concave function

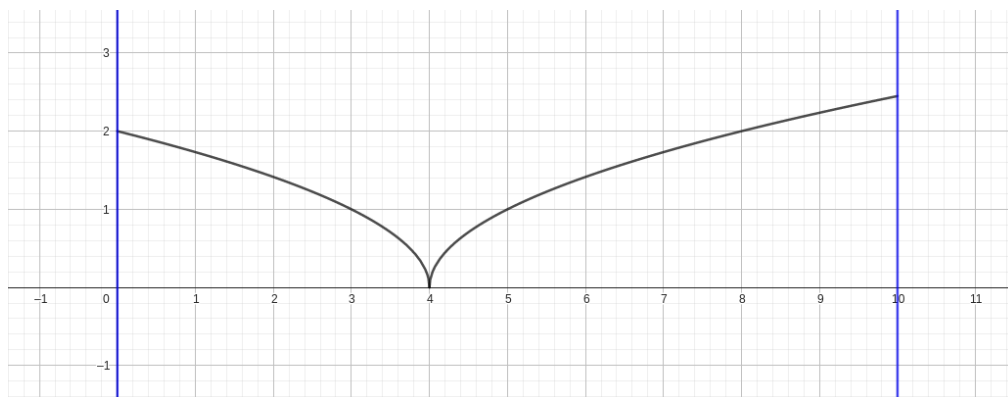


Figure 4: Example of a strictly quasiconvex function