

Handout 7 - ECON703 (Fall 2023)

1 Graph of a Function

Definition 1.1 (Graph of a Function). The graph of a function $f : X \rightarrow Y$ is defined as the set of ordered pairs $\Gamma(f) = \{(x, y) \mid f(x) = y\} \subset X \times Y$.

Definition 1.2 (Hypograph and Epigraph). The epigraph of a function $f : X \rightarrow \mathbb{R}$ is defined as

$$\text{Epi}(f) = \{(x, y) \mid y \geq f(x)\} \subset X \times \mathbb{R}$$

Its hypograph is defined as

$$\text{Hypo}(f) = \{(x, y) \mid y \leq f(x)\} \subset X \times \mathbb{R}$$

We can extend those definitions to functions whose codomain is a partially ordered set.

2 Isoquants and Contour Sets

Definition 2.1 (Isoquant). Let $f : X \rightarrow \mathbb{R}$ be a function from a set X to $\mathbb{R}$. Then the isoquant of f for the value $z \in \mathbb{R}$ is defined as

$$\{x \in X \mid f(x) = z\}$$

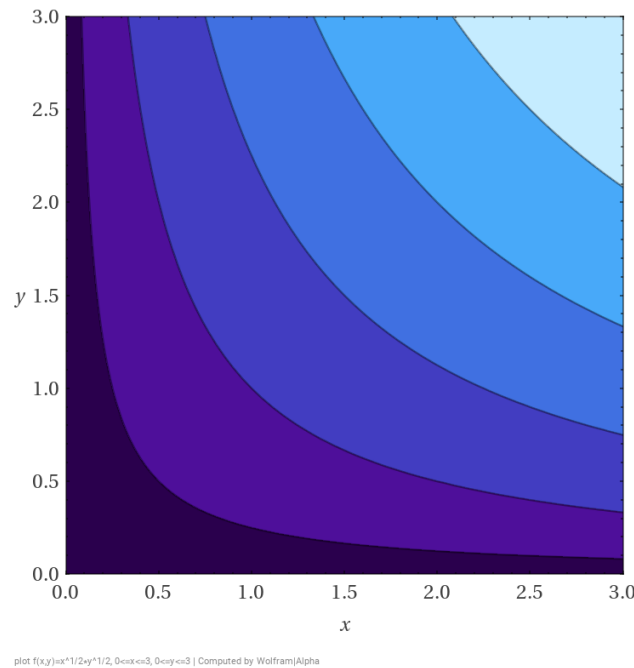


Figure 1: A contour plot of a Cobb-Douglas function

Definition 2.2 (Upper Contour Set). Let $f : X \rightarrow \mathbb{R}$ be a function from a set X to $\mathbb{R}$. Then the upper contour set of f at $z \in \mathbb{R}$ is defined as

$$\mathcal{C}_z^+ = \{x \in X \mid f(x) \geq z\}$$

Definition 2.3 (Lower Contour Set). Let $f : X \rightarrow \mathbb{R}$ be a function from a set X to $\mathbb{R}$. Then the lower contour set of f at $z \in \mathbb{R}$ is defined as

$$\mathcal{C}_z^- = \{x \in X \mid f(x) \leq z\}$$

Like before, we can extend these definitions to functions whose codomain is a partially ordered set.

3 Differentiability

Definition 3.1 (Directional Limits). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function from $\mathbb{R}$ to $\mathbb{R}$ and $I \subset \mathbb{R}$ be an interval. Let $a \in I$. We call $L \in \mathbb{R}$ the left limit of f at a if

$$\lim_{x \rightarrow a^-} f(x) = L \iff \forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall x \in I : (0 < a - x < \delta) \implies (|f(x) - L| < \epsilon)$$

We call $R \in \mathbb{R}$ the right limit of f at a if

$$\lim_{x \rightarrow a^+} f(x) = R \iff \forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall x \in I : (0 < x - a < \delta) \implies (|f(x) - R| < \epsilon)$$

Definition 3.2 (Differentiability in One Dimension). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is differentiable at $a \in \mathbb{R}$ if its derivative

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

exists.

Definition 3.3 (Differentiable Function in One Dimension). Let $U \subset \mathbb{R}$ be a subset of the real line. We say that $f : U \rightarrow \mathbb{R}$ is a differentiable function if it is differentiable at all $x \in U$.

Example 3.1 (Differentiable Function). Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$. Then f is a differentiable function.

Proof.

$$\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} 2x + h = 2x$$

Thus, the limit exists, and we even found the derivative. □

Example 3.2 (Nondifferentiable Function). Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 2x & \text{if } x \leq 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$$

Then f is not differentiable at $x = 2$.

Proof. We check sequences that approach 2 from the left and the right. In this case, these correspond to the left and right limits at $x = 2$.

Let $(h_n)_{n \in \mathbb{N}}$ be given by $h_n = 2^{-n}$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f(2 + h_n) - f(2)}{h_n} &= \lim_{n \rightarrow \infty} \frac{3(2 + h_n) - 2 - 4}{h_n} = \lim_{n \rightarrow \infty} \frac{3h_n}{h_n} = 3 \\ \lim_{n \rightarrow \infty} \frac{f(2 - h_n) - f(2)}{-h_n} &= \lim_{n \rightarrow \infty} \frac{2(2 - h_n) - 4}{-h_n} = \lim_{n \rightarrow \infty} \frac{-2h_n}{-h_n} = 2 \end{aligned}$$

Thus, we have shown that there are sequences converging to 2 such that the corresponding difference quotients converge to different values. Therefore, the limit, i.e., the derivative, does not exist at $x = 2$. $\square$

4 Cardinal vs. Ordinal Properties

Definition 4.1 (Ordinal Property). We say that a property of a function is ordinal if it is preserved under any strictly increasing transformation.

Example 4.1 (Quasiconcavity). Quasiconcavity is preserved under all strictly increasing transformations, as the upper and lower contour sets of a function do not change under strictly increasing transformations.

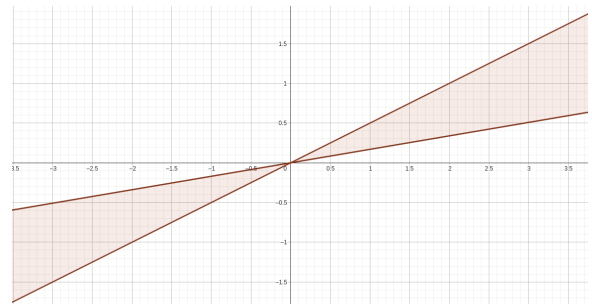
Definition 4.2 (Cardinal Property). A cardinal property of a function is a property that is not necessarily preserved under strictly increasing transformations.

Example 4.2 (Concavity). Consider the function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ given by $f(x) = \sqrt{x}$. Then f is concave. Consider the transformation $g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ given by $g(x) = x^4$, which is strictly increasing. Then $(g \circ f)(x) = x^2$ is a strictly increasing transformation of f that is not concave.

5 Homogeneous and Homothetic Functions

Definition 5.1 (Cone). Let $V \subset \mathbb{R}^n$ be a subset of Euclidean space. We call V a cone if

$$\forall x \in V \forall \lambda \in \mathbb{R}_{>0} : \quad \lambda x \in V$$



Example of a cone in $\mathbb{R}^2$

Definition 5.2 (Homogeneous Function). Let $V \subset \mathbb{R}^n$ be a cone and $f : V \rightarrow \mathbb{R}$ be a function from V to the real numbers. We say that f is a homogeneous function of degree $k \in \mathbb{R}$ if

$$\forall \lambda \in \mathbb{R}_{>0} \forall x \in V : \quad f(\lambda x) = \lambda^k f(x)$$

Definition 5.3 (Homothetic Function). We say a function f is homothetic if it is a strictly increasing transformation of a homogeneous function.

$\Rightarrow$ A homothetic function has isoquants that are radial rescalings of one another.