

Handout 10 - ECON703 (Fall 2023)

1 Linear Algebra

Definition 1.1 (Identity Matrix). Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. We call A the n -dimensional identity matrix, denoted $\mathbb{I}_n$, if

$$\forall i = 1, \dots, n: \quad A_{i,i} = 1 \quad \text{and} \quad \forall i \neq j: \quad A_{i,j} = 0$$

Definition 1.2 (Matrix Addition). Let A and B be two $m \times n$ matrices. Their sum is defined elementwise, i.e.

$$\forall i = 1, \dots, m \forall j = 1, \dots, n: \quad (A + B)_{i,j} = A_{i,j} + B_{i,j}$$

Definition 1.3 (Scalar Multiplication). Let A be an $m \times n$ matrix and $\lambda \in \mathbb{R}$ be a scalar. The scalar product of λ with A is defined elementwise, i.e.

$$\forall i = 1, \dots, m \forall j = 1, \dots, n: \quad (\lambda A)_{i,j} = \lambda A_{i,j}$$

Definition 1.4 (Matrix Multiplication). The product of two matrices A and B , denoted by AB , is defined if and only if the number of columns of A is equal to the number of rows of B . It is **NOT** commutative. Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$. Then the matrix product AB exists and is defined componentwise by

$$\forall i = 1, \dots, m \forall j = 1, \dots, p: \quad (AB)_{i,j} = \sum_{r=1}^n A_{i,r} B_{r,j}$$

This product is an $m \times p$ matrix.

There are other types of matrix products, for example, the Hadamard product and Kronecker product. These are used less frequently, but you will come across them, for example, in econometrics.

- [https://en.wikipedia.org/wiki/Hadamard_product_\(matrices\)](https://en.wikipedia.org/wiki/Hadamard_product_(matrices))
- https://en.wikipedia.org/wiki/Kronecker_product

Definition 1.5 (Transpose of a Matrix). The transpose of a matrix $A \in \mathbb{R}^{m \times n}$, denoted $A' \in \mathbb{R}^{n \times m}$, is defined elementwise by

$$\forall i = 1, \dots, n \forall j = 1, \dots, m: \quad (A')_{i,j} = A_{j,i}$$

and is an $n \times m$ matrix.

Definition 1.6 (Determinant of a Matrix). Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. Its determinant is defined via the Leibniz formula.

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n A_{i,\sigma(i)}$$

https://en.wikipedia.org/wiki/Leibniz_formula_for_determinants

In your first year, you will rarely (if at all) need to use determinants for matrices larger than 3×3 . So, we give the formulas for 2×2 and 3×3 matrices more directly.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

Definition 1.7 (Matrix Inverse). Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. If there exists some $B \in \mathbb{R}^{n \times n}$ such that $AB = BA = \mathbb{I}_n$, then we call B the matrix inverse of A and denote it by A^{-1} . It exists if and only if $\det(A) \neq 0$.

Definition 1.8 (Symmetric Matrix). We say a square matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if

$$\forall i = 1, \dots, n \quad \forall j = 1, \dots, n \quad A_{i,j} = A_{j,i}$$

This is equivalent to saying that $A = A'$.

Definition 1.9 (Positive / Negative (Semi-)Definite Matrix). We say a symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive (negative) definite if

$$\forall x \in \mathbb{R}^n \setminus \{0\} : \quad x'Ax > (<)0$$

We call it positive (negative) semidefinite if

$$\forall x \in \mathbb{R}^n \setminus \{0\} : \quad x'Ax \geq (\leq)0$$

Definition 1.10 (Trace). The trace of a square matrix $A \in \mathbb{R}^{n \times n}$, denoted $\text{tr}(A)$, is the sum of its entries on the main diagonal.

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}$$

Definition 1.11 (Eigenvectors and Eigenvalues). Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. Eigenvectors of A are non-zero vectors $\mu \in \mathbb{R}^n$ that satisfy the following property:

$$\exists \lambda \in \mathbb{R} \quad \text{s.t.} \quad A\mu = \lambda\mu$$

The λ corresponding to an eigenvector μ is called the corresponding eigenvalue. We typically consider eigenvectors whose length is normalized to one.

2 Linearity

Definition 2.1 (Linear Transformation). Let W and V be vector spaces over $\mathbb{R}$ and let $f : V \rightarrow W$ be a function between them. We say that f is a linear transformation or linear map, if

$$\begin{aligned} \forall u, v \in V : \quad & f(u + v) = f(u) + f(v) \\ \forall v \in V \quad \forall \lambda \in \mathbb{R} : \quad & f(\lambda v) = \lambda f(v) \end{aligned}$$

Example 2.1 (Matrix). Let $A \in \mathbb{R}^{m \times n}$ be a matrix and define the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$f(x) = Ax$$

Then f is a linear transformation.

Proof. Check the two properties with the definition of matrix multiplication given above. $\square$

3 Integration

The two integration rules we use most often during the first year are integration by parts and integration by substitution.

Theorem 3.1 (Integration by Parts).

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)] \Big|_a^b - \int_a^b u'(x)v(x)dx$$

Example 3.1 (Integration by Parts). Consider the following integral.

$$\int_0^2 xe^{2x}dx$$

Then, in our notation from before, we can set

$$\begin{aligned} u(x) = x &\implies u'(x) = 1 \\ v'(x) = e^{2x} &\implies v(x) = \frac{1}{2}e^{2x} \end{aligned}$$

Using integration by parts, this gives us the following

$$\int_0^2 xe^{2x}dx = \left[\frac{x}{2}e^{2x} \right]_0^2 - \int_0^2 \frac{1}{2}e^{2x}dx = e^4 - \left(\frac{1}{4}e^4 - \frac{1}{4} \right) = \frac{1}{4} + \frac{3}{4}e^4 \approx 41.199$$

Theorem 3.2 (Integration by Substitution). Let $g : [a, b] \rightarrow \mathbb{R}$ be a continuously differentiable function and $f : \mathbb{R} \rightarrow \mathbb{R}$ a continuous function. Then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 3.2 (Integration by Substitution). Consider the following integral.

$$\int_0^{\frac{\pi}{2}} (\sin(x))^2 \cos(x)dx$$

and set $g(x) = \sin(x)$. Then, by integration by substitution:

$$\int_0^{\frac{\pi}{2}} (\sin(x))^2 \cos(x)dx = \int_{\sin(0)}^{\sin(\frac{\pi}{2})} u^2 du = \int_0^1 u^2 du = \left[\frac{1}{3}u^3 \right]_0^1 = \frac{1}{3}$$

4 Taylor Approximation

Theorem 4.1 (Taylor's Theorem (Univariate, Peano Remainder)). Let $f \in \mathcal{C}^k(A, \mathbb{R})$ for a set $A \subset \mathbb{R}$. Let $x_0 \in \text{int}(A)$. Assume that f is k times differentiable at x_0 . Then there exists $h_k(x)$ with $\lim_{x \rightarrow x_0} h_k(x) = 0$ such that

$$f(x) = \sum_{i=0}^k \frac{1}{i!} f^{(i)}(x_0)(x - x_0)^i + h_k(x)(x - x_0)^k$$

We call the following term the k th-order Taylor polynomial:

$$P_k(x) = \sum_{i=0}^k \frac{1}{i!} f^{(i)}(x_0)(x - x_0)^i$$

and the other term the remainder term: $R_k(x) = h_k(x)(x - x_0)^k$.

We can give explicit formulas for the remainder term that can help us gauge our approximation's accuracy.

Theorem 4.2 (Lagrange-Form and Cauchy-Form of the Remainder). Let $f \in \mathcal{C}^k(A, \mathbb{R})$ for a set $A \subset \mathbb{R}$. Let $x_0 \in \text{int}(A)$. Assume that f is $k+1$ times differentiable between x and x_0 and that $f^{(k)}$ is continuous between x and x_0 . Then we can find the following explicit formula for the remainder term, called the Lagrange form:

$$\exists \xi_L \text{ between } x \text{ and } x_0 \quad \text{s.t.} \quad R_k(x) = \frac{f^{(k+1)}(\xi_L)}{(k+1)!} (x - x_0)^{k+1}$$

Similarly, we can find the following expression, called the Cauchy form:

$$\exists \xi_C \text{ between } x \text{ and } x_0 \quad \text{s.t.} \quad R_k(x) = \frac{f^{(k+1)}(\xi_C)}{k!} (x - \xi_C)^k (x - x_0)$$

Example 4.1 (First-Order Logarithm Approximation). Let $f(x) = \log(1+x)$. Then we can use the first Taylor polynomial around $x = 0$ to approximate the logarithm for small values of x as

$$\log(1+x) \approx \log(1) + x \frac{\partial}{\partial x} \log(1+x) \Big|_{x=0} = 0 + \frac{x}{1+0} = x$$