

Handout 11 - ECON703 (Fall 2023)

1 Midterm - Common Pitfalls

Assuming Interior Solutions:

Assume we face the following optimization problem:

$$\max_{x \in \mathbb{R}_{\geq 0}^n} u(x) \quad \text{s.t.} \quad x_i \geq 0 \quad \forall i = 1, \dots, n, \quad p'x \leq y$$

We can **NOT** automatically jump to equating marginal utilities!

$$\frac{MU_i}{p_i} = \frac{MU_j}{p_j} \quad \text{only applies if we consume both goods at the optimum}$$

If we do not consume good i , then $\frac{MU_i}{p_i}$ is no larger than the common marginal utility per dollar for any good consumed at the optimum.

Forgetting Lagrange Multipliers:

Similarly, if we set up a Lagrangian for an optimization problem, we need a Lagrange multiplier for **every** constraint. For the problem used in the previous paragraph:

$$\mathcal{L} = u(x) + \sum_{i=1}^n \lambda_i x_i + \lambda_y \left(y - \sum_{i=1}^n p_i x_i \right)$$

If we can somehow deduce that individual constraints will not bind, we can get rid of those multipliers, but we have to be careful.

Not checking whether a solution makes sense / is well-defined:

When you derive a solution, you should check whether it is well-defined (for example, it does not contain a square root of a negative number) and makes sense in the economic context (for example, no negative demand for a consumer good).

2 Midterm - Cost-Function Problem

Consider the following two production technologies:

$$\text{Technology 1: } \frac{1}{Q_1} = \frac{1}{K_1} + \frac{1}{L_1} \quad \text{and} \quad \text{Technology 2: } \sqrt{Q_2} = \sqrt{K_2} + \sqrt{L_2}$$

We are interested in the cost function of a producer with access to both technologies. The producer buys labor at a price $w > 0$ and rents capital at a price $r > 0$. For Technology 1, interpret zero output by the continuous extension $Q_1 = 0$ when no capital or no labor is used; for $Q_1 > 0$, the reciprocal equation is equivalent to $Q_1 = \frac{K_1 L_1}{K_1 + L_1}$.

In other words, we are interested in the solution of the following minimization problem.

$$\begin{aligned} c(Q; w, r) &= \min_{K_1, K_2, L_1, L_2, Q_1, Q_2} w(L_1 + L_2) + r(K_1 + K_2) \\ \text{s.t.} \quad &K_1 \geq 0, K_2 \geq 0, L_1 \geq 0, L_2 \geq 0, Q_1 \geq 0, Q_2 \geq 0, \quad Q_1 + Q_2 = Q \\ &Q_1 = \begin{cases} \frac{K_1 L_1}{K_1 + L_1} & \text{if } K_1 + L_1 > 0 \\ 0 & \text{if } K_1 + L_1 = 0 \end{cases}, \quad \sqrt{Q_2} = \sqrt{K_2} + \sqrt{L_2} \end{aligned}$$

Now, we could apply KKT directly, but it would be quite tedious. Instead, we can derive the cost functions for both technologies individually and use the fact that they are **constant returns to scale (CRS)**.

$$\begin{aligned} c_1(Q_1; w, r) &= \min_{K_1, L_1} wL_1 + rK_1 \quad \text{s.t.} \quad K_1 \geq 0, L_1 \geq 0, \quad \frac{1}{Q_1} = \frac{1}{K_1} + \frac{1}{L_1} \\ c_2(Q_2; w, r) &= \min_{K_2, L_2} wL_2 + rK_2 \quad \text{s.t.} \quad K_2 \geq 0, L_2 \geq 0, \quad \sqrt{Q_2} = \sqrt{K_2} + \sqrt{L_2} \end{aligned}$$

Since the technologies are **CRS**, the cost functions will take the form:

$$c_1(Q_1; w, r) = Q_1 c_1(1; w, r) \quad \text{and} \quad c_2(Q_2; w, r) = Q_2 c_2(1; w, r)$$

where $c_1(1; w, r) > 0$ and $c_2(1; w, r) > 0$ are constants that depend on the prices alone. Thus, the marginal cost of producing an additional unit of quantity is constant for both technologies. Thus, the only case in which it can be rational to use both technologies is if $c_1(1; w, r) = c_2(1; w, r)$. Otherwise, we will only use the technology associated with the smaller constant.

Therefore, we get the following relationship:

$$c(Q; w, r) = Q \min\{c_1(1; w, r), c_2(1; w, r)\}$$

Now calculate $c_1(1; w, r)$ and $c_2(1; w, r)$. Let $\rho_1 = -1$ and $\rho_2 = \frac{1}{2}$ and consider the following optimization problem.

$$\begin{aligned} c_i(Q_i; w, r) &= \min_{K_i, L_i} wL_i + rK_i \quad \text{s.t.} \quad K_i \geq 0, L_i \geq 0, \quad Q_i^{\rho_i} = K_i^{\rho_i} + L_i^{\rho_i} \\ &= \min_{K_i, L_i} wL_i + rK_i \quad \text{s.t.} \quad K_i \geq 0, L_i \geq 0, \quad Q_i = (K_i^{\rho_i} + L_i^{\rho_i})^{\frac{1}{\rho_i}} \end{aligned}$$

Let $\sigma_i = \frac{1}{1-\rho_i}$ and thus $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = 2$. Following John's CES cost function notes, we obtain the following relationship.

$$c_i(1; w, r)^{1-\sigma_i} = w^{1-\sigma_i} + r^{1-\sigma_i}$$

and thus

$$c_1(1; w, r) = \left(w^{\frac{1}{2}} + r^{\frac{1}{2}}\right)^2 \quad \text{and} \quad c_2(1; w, r) = (w^{-1} + r^{-1})^{-1} = \frac{wr}{w+r}$$

Now we need to determine which one is smaller.

$$\left(w^{\frac{1}{2}} + r^{\frac{1}{2}}\right)^2 > \frac{wr}{w+r} \iff \left(w^{\frac{1}{2}} + r^{\frac{1}{2}}\right)^2 - \frac{wr}{w+r} > 0 \iff \frac{r^2 + rw + w^2}{r+w} + 2\sqrt{rw} > 0$$

This holds if $w > 0$ and $r > 0$. Thus, the producer will use only the second technology. Therefore, the producer's cost function is $c(Q; w, r) = Q \frac{wr}{w+r}$.

3 Implicit Function Theorem

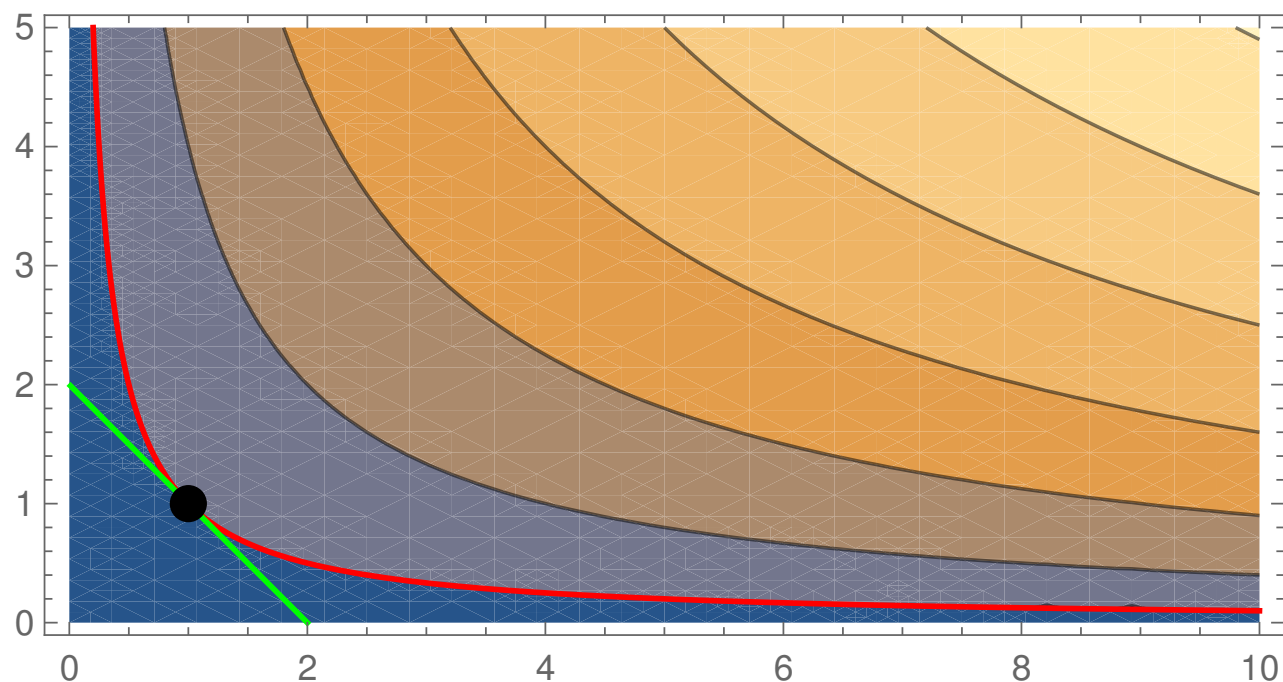
Theorem 3.1 (Implicit Function Theorem on Euclidean Spaces). Let $A \subset \mathbb{R}^{n+m}$ be open and let $f : A \rightarrow \mathbb{R}^m$ be a continuously differentiable function. Let $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+m}$ denote a point such that $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. Let $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{n+m}$ be a point such that $f(\mathbf{a}, \mathbf{b}) = 0$. Write the Jacobian of f as follows:

$$J_f(\mathbf{a}, \mathbf{b}) = \left[\begin{array}{ccc|ccc} \frac{\partial f_1}{\partial x_1}(\mathbf{a}, \mathbf{b}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}, \mathbf{b}) & \frac{\partial f_1}{\partial y_1}(\mathbf{a}, \mathbf{b}) & \dots & \frac{\partial f_1}{\partial y_m}(\mathbf{a}, \mathbf{b}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}, \mathbf{b}) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}, \mathbf{b}) & \frac{\partial f_m}{\partial y_1}(\mathbf{a}, \mathbf{b}) & \dots & \frac{\partial f_m}{\partial y_m}(\mathbf{a}, \mathbf{b}) \end{array} \right] = [X(\mathbf{a}, \mathbf{b}) | Y(\mathbf{a}, \mathbf{b})]$$

If $Y(\mathbf{a}, \mathbf{b})$ is invertible, then there are open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ with $\mathbf{a} \in U$, $\mathbf{b} \in V$, and $U \times V \subset A$ and a unique continuously differentiable function $g : U \rightarrow V$ such that $g(\mathbf{a}) = \mathbf{b}$ and $\forall \mathbf{x} \in U : f(\mathbf{x}, g(\mathbf{x})) = 0$. Moreover, the Jacobian of g for $\mathbf{x} \in U$ is given by

$$J_g(\mathbf{x}) = -[Y(\mathbf{x}, g(\mathbf{x}))]^{-1} X(\mathbf{x}, g(\mathbf{x}))$$

Example 3.1 (Iso-Utility Curves of a Cobb-Douglas Function). Consider $f : \mathbb{R}_{>0}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = \sqrt{xy} - 1$. This is a Cobb-Douglas utility function shifted to fit the setup of the implicit function theorem. At the point $(x, y) = (1, 1)$, we have $f(1, 1) = 0$.



The Jacobian of f is

$$J_f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \end{bmatrix}$$

Using the implicit function theorem, we can calculate the gradient of the iso-utility curve at $(1, 1)$. Let $g(x)$ be such that $f(x, g(x)) = 0$. In other words, let $g(x)$ describe the iso-utility curve for utility level 1.

Then

$$J_g(x) \Big|_{x=1, y=1} = - \left(\frac{\sqrt{x}}{2\sqrt{y}} \right)^{-1} \frac{\sqrt{y}}{2\sqrt{x}} \Big|_{x=1, y=1} = - \frac{y}{x} \Big|_{x=1, y=1} = -1$$

This shows that the slope of the indifference curve through $(1, 1)$ at $(1, 1)$ is -1 .