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Friedrich-Wilhelms-Universität Bonn

The effects of component permutation on estimated Impulse Responses in a three-dimensional SVAR[1] model

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Abgabedatum: 26.08.2019

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1 Introduction

In this paper, I am going to analyse the effects of changing the order of variables in a three-dimensional structural VAR[1] model on the impulse responses obtained by orthogonalizing the error terms in the VMA[∞] representation of the reduced form process. To do so, I am first going to explain the process of *Impulse Response Analysis* using Cholesky decomposition to orthogonalize the error terms and the importance of ordering the variables correctly.

To find out how bad the problems described theoretically are in reality, I will then perform a Monte Carlo simulation for a specific three-dimensional SVAR[1] process, where $\mathbf{B_0}$ is a lower triangular matrix. For the data generated by this process I am going to estimate VAR[1] models and their respective impulse response functions using Cholesky decomposition. I will then compare the estimated impulse response functions for different permutations of components to the theoretical impulse response functions and the estimated impulse response functions for the correct order. In my analysis I am also going to use confidence intervals for the estimated impulse responses in the correct order created via a bootstrap approach. These will be used to look at how often the impulse responses for incorrect permutations lie in those confidence intervals.

As a next step I am going to perform a similar analysis for a general SVAR[1] process, meaning that $\mathbf{B_0}$ does not have to be a lower triangular matrix. Here orthogonalization using a Cholesky decomposition is not economically justified in any order of components. I will look at how different the results are, if I use the method nevertheless and compare them to the theoretical impulse response function which can be calculated if $\mathbf{B_0}$ is known.

In the end, I am going summarize my findings and give an outlook on the process of identifying $\mathbf{B_0}$ and other methods of conducting impulse response analysis, that either deal with the problem of ordering the variables in another way or use a completely different approach to finding the impulse responses.

2 Impulse Response Analysis

2.1 Data Generating Process and its representations

In a multidimensional setting it is often of interest to know the response of one variable to an impulse in another variable. The process of analysing these responses is called *Impulse Response Analysis*. In this explanation of the general problem of changing the order of components when analysing the impulse responses of a specific process, I am going to follow structure and notation from Kilian and Lütkepohl 2017, and Lütkepohl 2007.

Consider a stationary three-dimensional time series y_t, $t = 1, ..., T$. I assume that the data generating process is an unknown structural vector autoregressive process of order one as shown in equation 1.

$$\mathbf{B_0} y_t = \mathbf{B_1} y_{t-1} + \omega_t, \quad y_t, \omega_t \in \mathbb{R}^3, \mathbf{B_0}, \mathbf{B_1} \in \mathbb{R}^{3\times 3} \tag{1}$$

Where ω_t is a serially uncorrelated error term with $\mathbb{E}(\omega_t) = 0$ and mutually uncorrelated elements. Therefore Σ_ω, the variance-covariance matrix of ω_t, is a diagonal matrix, which in the following will be assumed to be $\mathbf{I_3}$.[1]

Multiplying from the left by ${\mathbf{B_0}}^{-1}$ results in the reduced form of our data generating process, which can be estimated consistently. I will assume the reduced form to be known in the later parts of this chapter.

$$\begin{aligned} y_t &= {\mathbf{B_0}}^{-1} \mathbf{B_1} y_{t-1} + {\mathbf{B_0}}^{-1} \omega_t \\ &= \mathbf{A_1} y_{t-1} + \epsilon_t \end{aligned} \tag{2}$$

The reduced form can be written as a VMA[∞] process, by inserting recursively for $y_{t-i} \quad \forall i \in \mathbb{N}$. This yields a representation as an infinite sum of serially uncorrelated error terms.

$$\begin{aligned} y_t = \mathbf{A_1} y_{t-1} + \epsilon_t &= \mathbf{A_1}[\mathbf{A_1} y_{t-2} + \epsilon_{t-1}] + \epsilon_t \\ &= \mathbf{A_1}[\mathbf{A_1}[\mathbf{A_1} y_{t-3} + \epsilon_{t-2}] + \epsilon_{t-1}] + \epsilon_t = ... \\ &= \sum_{i=0}^{\infty} \Phi_i \epsilon_{t-i} \end{aligned} \tag{3}$$

This form is known as the VMA[∞] representation of the reduced form process. Since Σ_ϵ is not generally a diagonal matrix, meaning that the components of ϵ_t might be correlated, one cannot analyse the effect of an isolated impulse in one variable yet. First one has to transform the process in such a way, that the components of the error term are uncorrelated. This can be achieved via a so called Cholesky decomposition.

[1] This does not reduce generality as long as one does not impose restrictions on the diagonal elements of $\mathbf{B_0}$.

2.2 Cholesky Decomposition

Since Σ_ϵ is a variance-covariance matrix, it is Hermitian and positive semi-definite. I am going to assume it to be positive definite in the following, but a similar argument can be made for the generalized case.[2] Assuming positive definiteness, there is a unique lower triangular matrix P such that $\Sigma_\epsilon = PP^T$. This is the so called Cholesky decomposition of Σ_ϵ. Using this method, the process can be written as follows

$$y_t = \sum_{i=0}^{\infty} \Phi_i \epsilon_{t-i} = \sum_{i=0}^{\infty} \Theta_i \eta_{t-i} \quad \Theta_i = \Phi_i P \quad \eta_t = P^{-1}\epsilon_t \tag{4}$$

Σ_η is derived by

$$\begin{aligned} \Sigma_\eta &= \mathbb{E}[(\eta_t - \mu_\eta)(\eta - \mu_\eta)^T] = \mathbb{E}[\eta\eta^T] \\ &= \mathbb{E}[(P^{-1}\epsilon)(P^{-1}\epsilon)^T] = \mathbb{E}[P^{-1}\epsilon\epsilon^T P^{-1T}] \\ &= P^{-1}\mathbb{E}[\epsilon\epsilon^T]P^{-1T} = P^{-1}\Sigma_\epsilon P^{-1T} \\ &= P^{-1}PP^T P^{-1T} = \mathbf{I_3} \end{aligned} \tag{5}$$

As the components of η_t are uncorrelated, I am now able to analyse the impact of an isolated impulse in one component using equation 4. Assuming an impulse $\eta_t = (\eta_{1,t}, \eta_{2,t}, \eta_{3,t})^T$ the impulse responses can be written as

$$\frac{\partial y_{t+i}}{\partial \eta_t^T} = \Theta_i, \quad i = 0, 1, ..., H \tag{6}$$

where $\Theta_i \in \mathbb{R}^{3\times 3}$ is a matrix that describes the responses of each variable to each shocked component and H is the maximum propagation horizon of the impulse. In the later parts of this paper I chose $H = 10$ when portraying theoretical or estimated impulse response functions. Up until now I assumed the order of components to be correct in the reduced form, but since there are multiple possible permutations it is not assured that one would construct the process in this way, as from the data alone we cannot deduce the correct order.

2.3 Why the order of components is of importance

If one assumes $\mathbf{B_0}$ to be a lower triangular matrix, one such P for which $\Sigma_\epsilon = PP^T$ is $\mathbf{B_0}^{-1}$.

$$\begin{aligned} \Sigma_\epsilon &= \mathbb{E}[\epsilon_t\epsilon_t^T] = \mathbb{E}[(\mathbf{B_0}^{-1}\omega_t)(\mathbf{B_0}^{-1}\omega_t)] \\ &= \mathbb{E}[\mathbf{B_0}^{-1}\omega_t\omega_t^T\mathbf{B_0}^{-1T}] = \mathbf{B_0}^{-1}\mathbb{E}[\omega_t\omega_t^T]\mathbf{B_0}^{-1T} \\ &= \mathbf{B_0}^{-1}\mathbf{I_3}\mathbf{B_0}^{-1T} = \mathbf{B_0}^{-1}\mathbf{B_0}^{-1T} \end{aligned} \tag{7}$$

[2]For a positive semi-definite matrix Σ_ϵ the Cholesky decomposition does not have to be unique.

Since the Cholesky decomposition ist unique, it is possible to identify $\mathbf{B_0}$, if the components of the reduced form model are in correct order. So the theoretical impulse responses can be described using

$$y_t = \sum_{i=0}^{\infty} \Theta_i \eta_{t-i} \quad \Theta_i = \Phi_i \mathbf{B_0}^{-1} \quad \eta_t = \mathbf{B_0} \epsilon_t = \omega_t \tag{8}$$

Taking another look at equation 3 shows that $\Phi_i = A_1^i$. Replacing the corresponding parts in equation 8 results in equation 9.

$$\frac{\partial y_{t+i}}{\partial \eta_t^T} = \Theta_i = \mathbf{A_1}^{\mathbf{i}} \mathbf{B_0}^{\mathbf{-1}} = (\mathbf{B_0}^{\mathbf{-1}} \mathbf{B_1})^{\mathbf{i}} \mathbf{B_0}^{\mathbf{-1}} \tag{9}$$

The assumption that $\mathbf{B_0}$ is a lower triangular matrix gives the underlying model a particular structure and thereby implies a particular causal chain between the components of the process. Processes that share these qualities are called *Recursively Identified.*

$$\begin{aligned}
\mathbf{B_0} y_t &= \begin{bmatrix} b_{1,1}^0 & 0 & 0 \\ b_{2,1}^0 & b_{2,2}^0 & 0 \\ b_{3,1}^0 & b_{3,2}^0 & b_{3,3}^0 \end{bmatrix} \begin{bmatrix} Y_{1,t} \\ Y_{2,t} \\ Y_{3,t} \end{bmatrix} = \begin{bmatrix} b_{1,1}^0 Y_{1,t} \\ b_{2,1}^0 Y_{1,t} + b_{2,2}^0 Y_{2,t} \\ b_{3,1}^0 Y_{1,t} + b_{3,2}^0 Y_{1,t} + b_{3,3}^0 Y_{1,t} \end{bmatrix} \\
&= \begin{bmatrix} b_{1,1}^1 & b_{1,2}^1 & b_{1,3}^1 \\ b_{2,1}^1 & b_{2,2}^1 & b_{2,3}^1 \\ b_{3,1}^1 & b_{3,2}^1 & b_{3,3}^1 \end{bmatrix} \begin{bmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ Y_{3,t-1} \end{bmatrix} + \begin{bmatrix} \omega_{1,t} \\ \omega_{2,t} \\ \omega_{3,t} \end{bmatrix}
\end{aligned} \tag{10}$$

As can be seen from equation 10, Y_1 has a contemporaneous effect on Y_2 and Y_3, Y_2 has a contemporaneous effect on Y_3 and Y_3 does not have a contemporaneous effect on the other components of the process. This information cannot be found in the data without external restrictions on the elements of $\mathbf{B_0}$. I will drop the assumption of triangularity later, to study the effect of orthogonalizing the error terms on the impulse responses if it is not economically justified to assume recursive identification, i.e. if $\mathbf{B_0}$ is not a triangular matrix in any permutation of components.

If one continues to assume triangularity for $\mathbf{B_0}$ it is interesting in which way a change in the order of components of our reduced form models influences the series of matrices $[\Theta_i]$, since the orthogonalization of error terms now implies a different causal chain.

Let y_t^{xyz} denote the component permutation of y_t, such that the components are now in order x-y-z

$$y_t = \begin{bmatrix} Y_{1,t} \\ Y_{2,t} \\ Y_{3,t} \end{bmatrix} \quad y_t^{xyz} = \begin{bmatrix} Y_{x,t} \\ Y_{y,t} \\ Y_{z,t} \end{bmatrix} \tag{11}$$

and $\mathbf{A^{xyz}}$ denote the permutation of matrix $\mathbf{A}$ where rows and columns have been interchanged such that equation 12 is fulfilled.

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \quad \mathbf{A^{xyz}} = \begin{bmatrix} a_{x,x} & a_{x,y} & a_{y,z} \\ a_{y,x} & a_{y,y} & a_{x,z} \\ a_{z,x} & a_{z,y} & a_{z,z} \end{bmatrix} \tag{12}$$

The correct underlying model can be seen in equation 10. But if one uses y_t^{213} as the basis of the reduced form model permutate and orthogonalizes via a Cholesky decomposition, this implies a structure as shown below in equation 13.

$$\mathbf{\tilde{B}_0} y_t^{213} = \begin{bmatrix} \tilde{b}^0_{1,1} & 0 & 0 \\ \tilde{b}^0_{2,1} & \tilde{b}^0_{2,2} & 0 \\ \tilde{b}^0_{3,1} & \tilde{b}^0_{3,2} & \tilde{b}^0_{3,3} \end{bmatrix} y_t^{213} = \begin{bmatrix} \tilde{b}^1_{1,1} & \tilde{b}^1_{1,2} & \tilde{b}^1_{1,3} \\ \tilde{b}^1_{2,1} & \tilde{b}^1_{2,2} & \tilde{b}^1_{2,3} \\ \tilde{b}^1_{3,1} & \tilde{b}^1_{3,2} & \tilde{b}^1_{3,3} \end{bmatrix} y_{t-1}^{213} + \begin{bmatrix} \omega_{2,t} \\ \omega_{1,t} \\ \omega_{3,t} \end{bmatrix} \tag{13}$$

This structure implies that Y_2 has a contemporaneous effect on Y_1 and Y_3, that Y_1 has a contemporaneous effect on Y_3 and Y_3 to not have any contemporaneous effects on the other components, which constitutes a direct contradiction to the actual causal structure and is therefore a misspecification of the model. This means that impulse responses obtained via orthogonalization of the error term are different from the correct ones as the Cholesky decomposition now delivers a $\tilde{P} \neq (\mathbf{B_0^{213}})^{-1}$. Since one only knows the reduced form and cannot be certain, which permutation of components correctly identifies the impulse responses, this is very problematic.

The theoretical impulse response functions for the incorrect permutations can be found analogously to the correct order. To do so, I calculate the reduced form of our correctly specified model in an incorrect order.

$$\mathbf{B_0^{213}} y_t^{213} = \mathbf{B_1^{213}} y_{t-1}^{213} + \omega_t^{213} \tag{14}$$

where

$$\mathbf{B_0^{213}} = \begin{bmatrix} b^0_{2,2} & b^0_{2,1} & 0 \\ 0 & b^0_{1,1} & 0 \\ b^0_{3,2} & b^0_{3,1} & b^0_{3,3} \end{bmatrix} \quad \mathbf{B_1^{213}} = \begin{bmatrix} b^1_{2,2} & b^1_{2,1} & b^1_{2,3} \\ b^1_{1,2} & b^1_{1,1} & b^1_{1,3} \\ b^1_{3,2} & b^1_{3,1} & b^1_{3,3} \end{bmatrix} \quad \omega_t^{213} = \begin{bmatrix} \omega_{2,t} \\ \omega_{1,t} \\ \omega_{3,t} \end{bmatrix}$$

Using the notation introdued in equations 11 and 12, the reduced form can be written as

$$y_t^{213} = (\mathbf{B_0^{213}})^{-1} \mathbf{B_1^{213}} y_{t-1}^{213} + (\mathbf{B_0^{213}})^{-1} \omega_t^{213} \tag{15}$$

which in turn can be transformed into its VMA[∞] representation and then orthogonalised using a Cholesky decomposition

$$\begin{aligned} y_t^{213} &= \sum_{i=0}^{\infty} \Phi_i^{213} \epsilon_{t-i}^{213} = \sum_{i=0}^{\infty} ((\mathbf{B_0^{213}})^{-1} \mathbf{B_1^{213}})^i \tilde{P} \tilde{P}^{-1} \epsilon_{t-i}^{213} \\ &= \sum_{i=0}^{\infty} ((\mathbf{B_0^{213}})^{-1} \mathbf{B_1^{213}})^i \tilde{P} \eta_{t-i}^{213} \end{aligned} \tag{16}$$

Even though equation 16 is also a representation of our data generating process in the form of an infinite sum of error terms that are uncorrelated componentwise as in equation 8, this model misspecifies the underlying causal chain. For this permutation of components one obtains a different set of impulse response functions, since

$$\begin{aligned} \frac{\partial y_{t+i}^{213}}{\partial (\eta_t^{213})^T} &= ((\mathbf{B_0^{213}})^{-1} \mathbf{B_1^{213}})^i \tilde{P} \\ &\neq \Theta_i^{213} = [(\mathbf{B_0}^{-1} \mathbf{B_1})^i \mathbf{B_0}^{-1}]^{213} = ((\mathbf{B_0^{213}})^{-1} \mathbf{B_1^{213}})^i (\mathbf{B_0^{213}})^{-1} \end{aligned} \tag{17}$$

as $\tilde{P} \neq (\mathbf{B_0^{213}})^{-1}$. In the appendix I showed that $\mathbf{A^{213}B^{213}} = [\mathbf{AB}]^{\mathbf{213}}$, which equation 17 relies on. The statement holds true for other permutations as well, as can be shown analogously.

From this theoretical analysis I conclude that the usage of an incorrect order of variables in the process of analysing the impulse responses can be problematic. But since I cannot give a general formula for $\tilde{P}$ in each permutation, I do not know how far off the impulse response functions derived from an incorrect permutation lie from the theoretical impulse response function of the data generating process.

If $\mathbf{B_0}$ is not a lower triangular matrix most of these results can be constructed analogously, the difference being that there is no correct way to use Cholesky decomposition as a means to finding the impulse response functions. Since the model is not recursively identified it is not possible to obtain the correct sequence $[\Theta_i]$ as it was shown in equation 8 because $\mathbf{B_0}$ cannot be identified. The theoretical impulse response functions can still be constructed and for each permutation one can find estimated impulse response functions using the same method, but there is no reason to assume that these are similar to the theoretical ones.

To get a feeling for the effects of the problem which I described theoretically, I am going to perform Monte Carlo simulations analysing the differences in the generated impulse response functions in chapters 3 and 4.

3 Simulation for B_0 lower triangular matrix

3.1 Process and Simulation

In this section I will analyse the effects of choosing an incorrect order when orthogonalizing the error terms on the impulse response functions when B_0 is a lower triangular matrix. To do so, I am going to use a Monte Carlo simulation for a specific data generating process. Consider the stationary three-dimensional structural VAR[1] process

$$\mathbf{B_0} y_t = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1.5 & 2 \end{bmatrix} y_t = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.6 & 0.3 \\ 1.2 & 1.5 & 1.2 \end{bmatrix} y_{t-1} + \omega_t \tag{18}$$

where $\Sigma_\omega = \mathbf{I_3}$ and the other conditions mentioned in chapter 2 are fulfilled. I created 100 realisations of length 1000 for this process using its reduced form

$$y_t = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ -1 & 0.4 & 0.1 \\ 0.95 & 0.4 & 0.475 \end{bmatrix} y_{t-1} + \epsilon_t \quad \Sigma_\epsilon = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 5 & -2.75 \\ 1 & -2.75 & 1.8125 \end{bmatrix} \tag{19}$$

A realisation of this process can be seen in the appendix in figure 9. The process is stationary which can be shown by calculating the roots of its characteristic polynomial, its resulting variance-covariance matrix Σ_ϵ is positive definite. Its theoretical impulse response functions can be derived by transforming the reduced form into its VMA[∞] representation and then using the results from equation 8 and 9. These theoretical impulse response functions are depicted in figure 1.

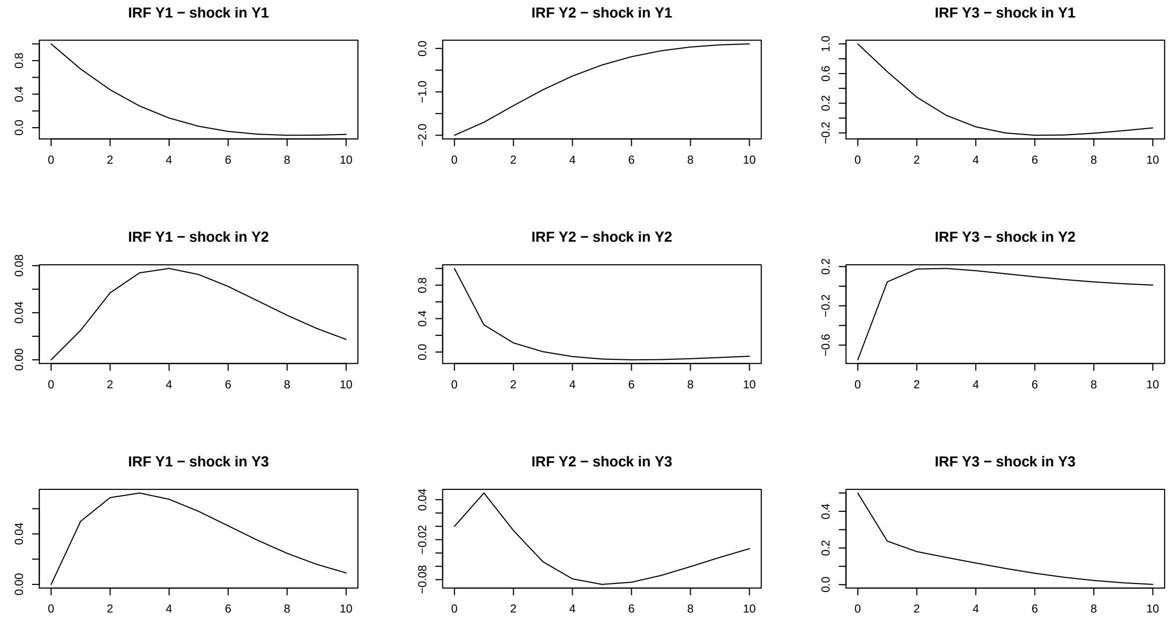

Figure 1: Theoretical Impulse Response Functions

The next step was to fit a VAR[1] model to each realisation of the time series in every permutation of components. This was done using the function *VAR* from the package *vars* in R.[3] Information about the implementation of this package can be found in Pfaff 2008. The model fitted to the realisation of the data generating process shown in figure 9 in its correct order is

$$\hat{y}_t = \begin{bmatrix} 0.016 \\ -0.046 \\ 0.002 \end{bmatrix} + \begin{bmatrix} 0.843 & 0.101 & 0.057 \\ -1.049 & 0.401 & 0.198 \\ 0.988 & 0.418 & 0.439 \end{bmatrix} y_{t-1} + e_t \tag{20}$$

3.2 Results

Comparing this estimated process to equation 19, the estimated values are quite close to the theoretical ones, which was expected, since the reduced form model can be estimated consistently from the data. For each of those estimated processes, I created 5 additional processes that were identical aside from the order of components. I did this to be able to compare the resulting estimates for the impulse response functions, which were calculated using the function *irf* from the same package. As shown in equation 17, the impulse response functions for different orderings of components do not have to be identical. For the correct permutation of components I calculated the estimated impulse response functions and a 95% confidence interval, which was constructed using a bootstrap approach. These estimates including the confidence interval for the data from figure 9 are shown in figure 2.

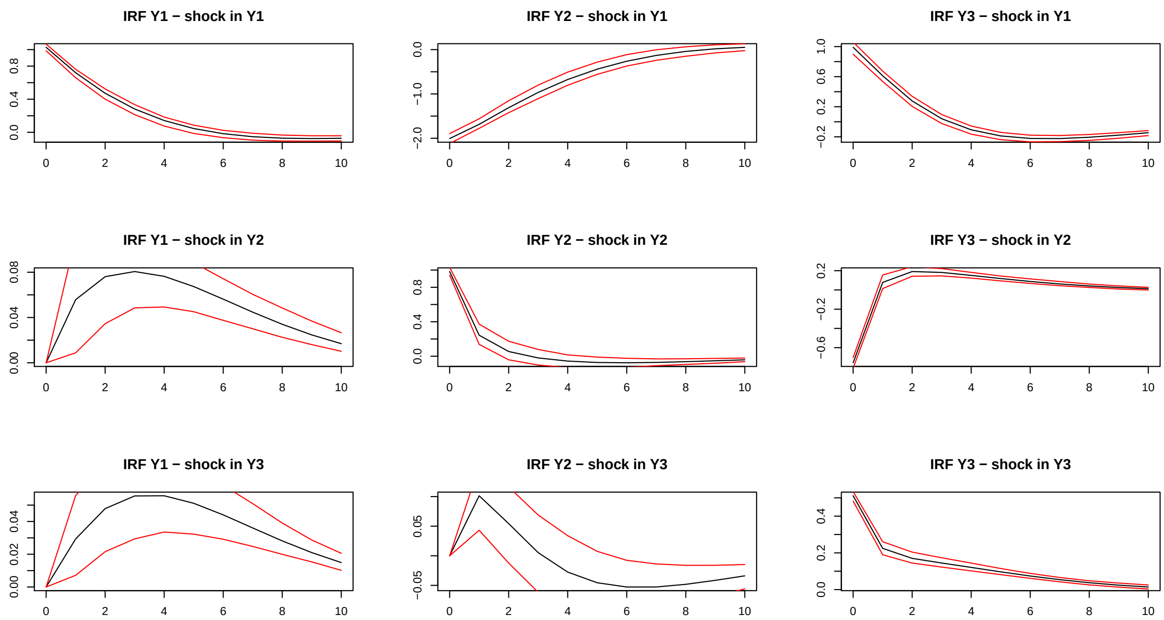

Figure 2: Estimated Impulse Response Functions including bootstrap CI

[3]https://cran.r-project.org/web/packages/vars/index.html

The function *irf* uses the same approach to finding the impulse response functions as I have described in 3.1, so it assumes that an orthogonalization in the given order is appropriate and orthogonalizes the error term using a Cholesky decomposition.

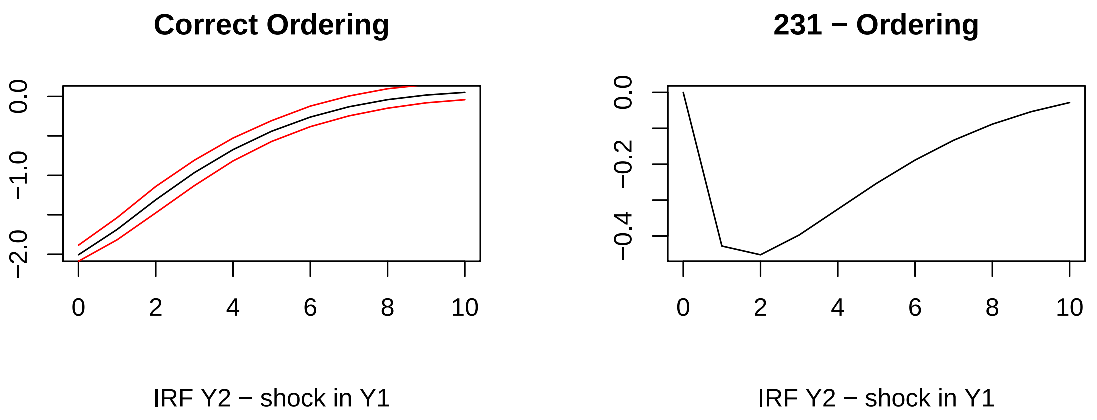

Figure 3: Comparison of two IRFs from different Permutations

As shown before orthogonalization of errors is problematic when the components of our time series are in an incorrect order. These differences can be seen in figure 3, which shows two different impulse response function for Y_2 to a shock in Y_1 and figure 4, which compares the means for the correct permutation (blue), the confidence intervals (red) and the incorrect permutations (black).

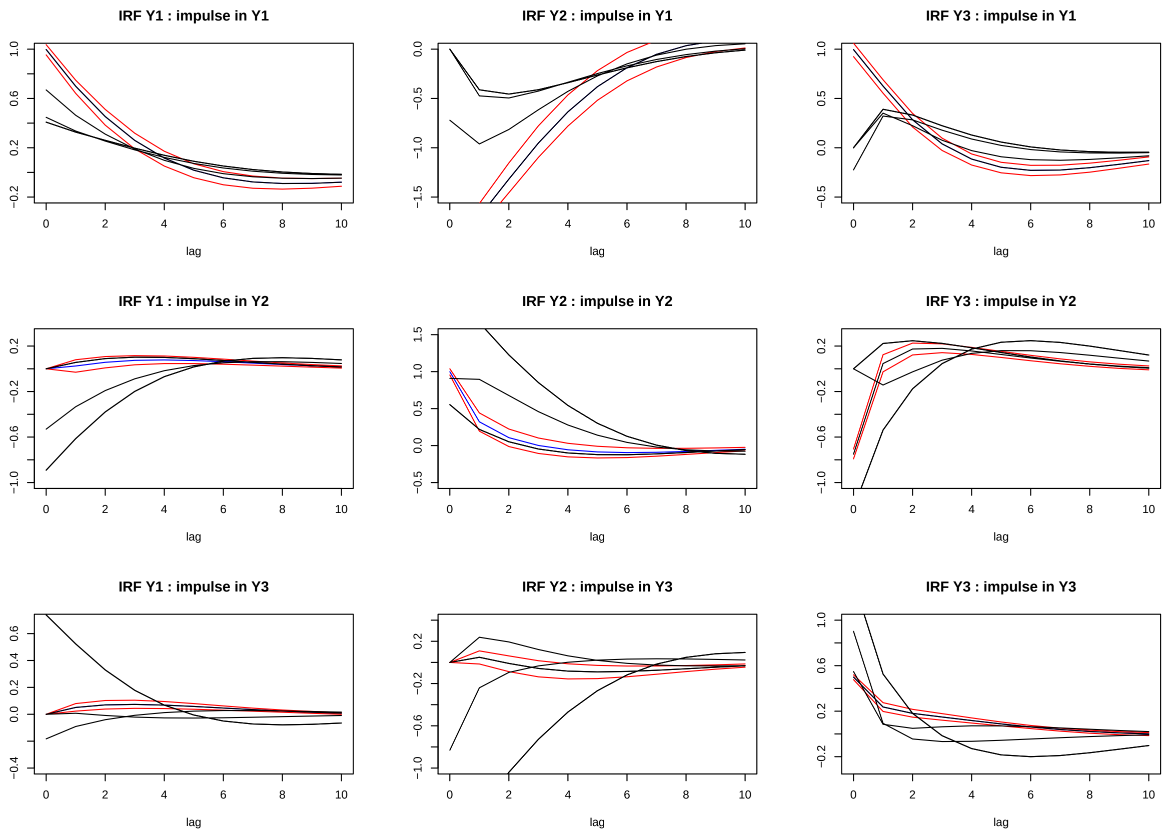

Figure 4: Means of IRFs in different permutations

From figure 4 alone, one can see that the estimated impulse responses show strong differences in their patterns and characteristics. Quite often the estimated responses for incorrect permutations lie outside of the confidence intervals and far from the estimates for the correct order of components or the theoretical values. Now I look at how often the estimated impulse response functions for different permutations lie in the respective confidence interval for the correct permutation of the same realisation of the time series. The results are shown in figure 5.

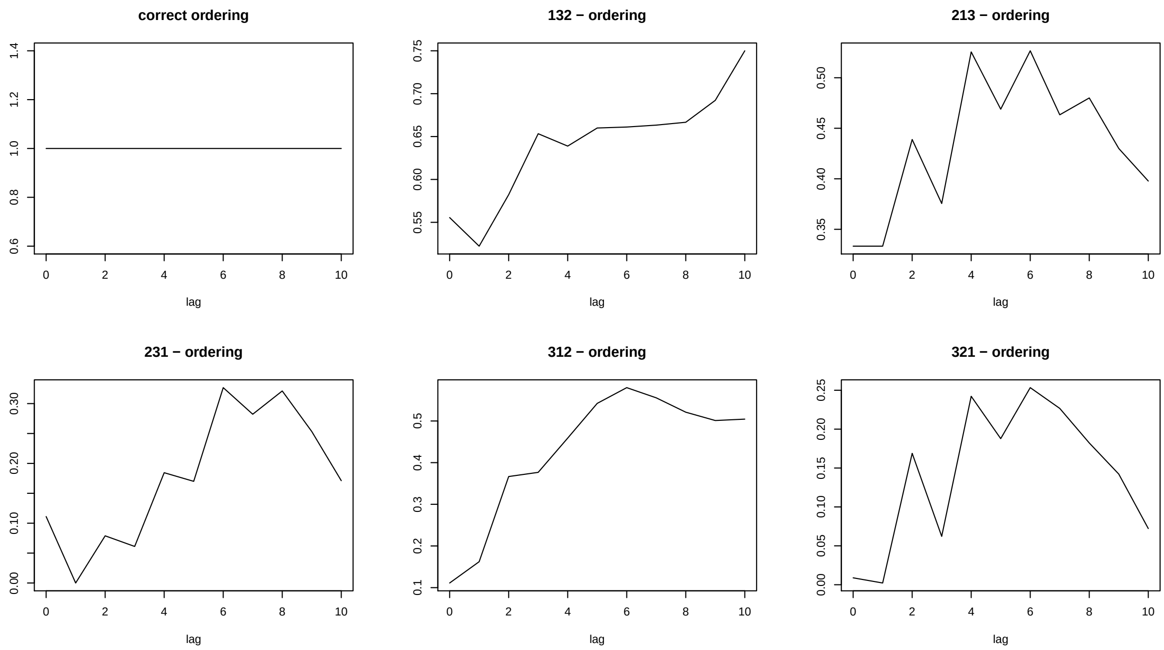

Figure 5: Fraction of estimated IRFs that lie in the bootstrap CI

As can be seen in figure 5 the impulse response functions for some permutations like 2-3-1 or 3-2-1 lie in the confidence interval quite rarely. Because these values are calculated for individual lags, one can use the lowest point in each graph as an upper boundary for the fraction of estimated impulse response functions that lie completely in the constructed confidence interval. These upper boundaries and the calculated values can be seen in the following table.

Permutation	correct	1-3-2	2-1-3	2-3-1	3-1-2	3-2-1
Upper boundary	1	0.75	0.527	0.327	0.58	0.253
Calculated fraction	1	0.406	0.333	0	0.072	0

As can be seen in the table, in reality the fraction of estimated impulse responses that lie in the confidence interval completely is even lower. Since one cannot know the correct order of variables just from the data, there is no certainty, which impulse response functions are the ones one should work with. To choose a permutation of components one has to think about the economical mechanism that drives the time series and derive restrictions on $\mathbf{B_0}$ to obtain the necessary information.

Taking a closer look at the impulse response functions generated for the correct permutation and permutation 1-3-2 I find, that they are in fact identical for shocks in component Y_1 for every realisation of our data generating process. This can be explained by looking at how those impulse response functions are derived. Starting with equation 17 and adapting it for order 1-3-2, I obtain

$$\begin{aligned} \frac{\partial y_{t+i}^{132}}{\partial(\eta_t^{132})^T} &= ((\mathbf{B_0^{132}})^{-1}\mathbf{B_1^{132}})^i \tilde{P} \\ &\neq \Theta_i^{132} = [(\mathbf{B_0}^{-1}\mathbf{B_1})^i \mathbf{B_0}^{-1}]^{132} = ((\mathbf{B_0^{132}})^{-1}\mathbf{B_1^{132}})^i (\mathbf{B_0^{132}})^{-1} \end{aligned} \tag{21}$$

If one now looks at $\tilde{P}$, which is the result of using a Cholesky decomposition on the process in order 1-3-2, it becomes clear that its first row is identical to the first row of $(\mathbf{B_0^{132}})^{-1}$ where the second and third column have been exchanged. Similar patterns can be found for other permutations relative to each other.

Combining the information from figure 4 and figure 5, I can conclude that misspecificating the order of components is in fact a serious problem, if one wants to conduct impulse response analysis on data generated by the process in equation 18. So even though some permutations might yield correct impulse responses for specific components in case of specific impulses, these findings show that one has to be careful and consider the economic theory behind the modelled time series when using orthogonalization as a means of finding the impulse response functions, even if it is reasonable to assume recursive identification in some permutation.

4 Simulation for general B_0

4.1 Process and Simulation

Now consider the case where $\mathbf{B_0}$ is not a lower triangular matrix. In this case orthogonalizing the error term using a Cholesky decomposition is not a valid approach to analyse the impulse responses, since the model is not recursively identified. As said before orthogonalization is only advisable if it is economically justified to assume a particular causal chain in the underlying process. In this chapter I am going to study the effect of using the aforementioned method if it is not possible to do so. Consider the following data generating process

$$\mathbf{B_0} y_t = \begin{bmatrix} 0.6 & 0.9 & 0.2 \\ 0.3 & 0.2 & 1 \\ 1.1 & 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} Y_{1,t} \\ Y_{2,t} \\ Y_{3,t} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.5 & 0.5 & 0.2 \\ 0.8 & 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ Y_{3,t-1} \end{bmatrix} + \begin{bmatrix} \omega_{1,t} \\ \omega_{2,t} \\ \omega_{3,t} \end{bmatrix} \tag{22}$$

This model is not recursively identified as $\mathbf{B_0}$ is not a lower triangular matrix. Its reduced form is constructed analogously to the special case discussed in chapter 3.

$$y_t = \begin{bmatrix} 0.537 & -0.068 & -0.127 \\ 0.244 & 0.043 & 0.382 \\ 0.29 & 0.512 & 0.162 \end{bmatrix} y_{t-1} + \epsilon_t \tag{23}$$

where

$$\Sigma_\epsilon = \mathbf{B_0}^{-1}\mathbf{B_0}^{-1T} = \begin{bmatrix} 2.416 & -2.265 & -0.541 \\ -2.265 & 3.298 & -0.043 \\ -0.541 & -0.043 & 1.265 \end{bmatrix}$$

This process is stationary as can be shown by checking the roots of its characteristic polynomial. Its resulting variance-covariance matrix Σ_ϵ is positive definite.

As shown in equation 8 and 9 before, I am able to calculate the theoretical impulse responses using the VMA[∞] representation of the reduced form process if $\mathbf{B_0}$ is known. This is possible since it is not necessary for $\mathbf{B_0}$ to be a lower diagonal matrix to use this approach. The theoretical impulse response functions can be seen in figure 6.

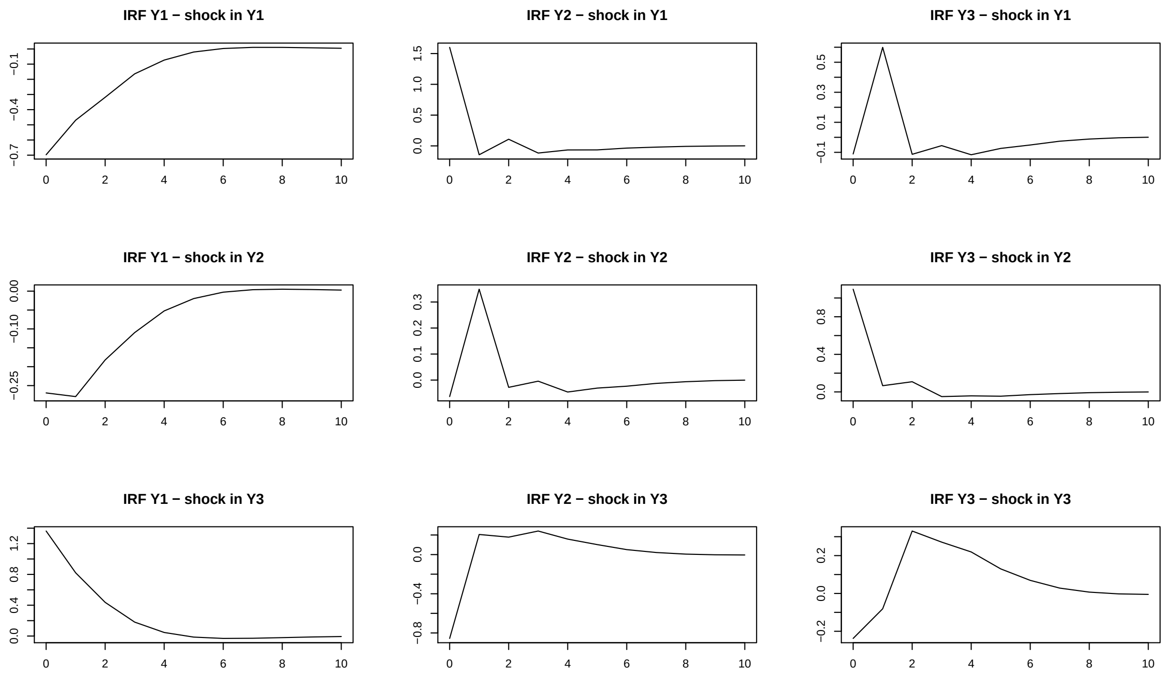

Figure 6: Theoretical Impulse Response Functions for the general process

As in chapter 3, I created 100 realisations of length 1000 using the reduced form described in equation 23. One realisation is shown in figure 10 in the appendix. For each of those data sets, I fitted a VAR[1] model using the same packages and functions as in chapter 3. The fitted model for the data depicted in figure 10 is shown in equation 24.

$$\hat{y}_t = \begin{bmatrix} 0.022 \\ -0.0007 \\ 0.042 \end{bmatrix} + \begin{bmatrix} 0.552 & -0.016 & -0.136 \\ 0.192 & -0.009 & 0.392 \\ 0.322 & 0.494 & 0.164 \end{bmatrix} y_{t-1} + e_t \tag{24}$$

For every one of those models I calculated the impulse response functions in each order of components analogously to chapter 3, even though orthogonalization is not economically justified in any permutation of components. So none of the sets of impulse response functions calculated this way should describe the process correctly as this method is not able to identify $\mathbf{B_0}$.

4.2 Results

Looking at the estimated impulse responses we can see a multitude of things. The means of the impulse response functions created for these permutations over all realisations are shown in figure 7.

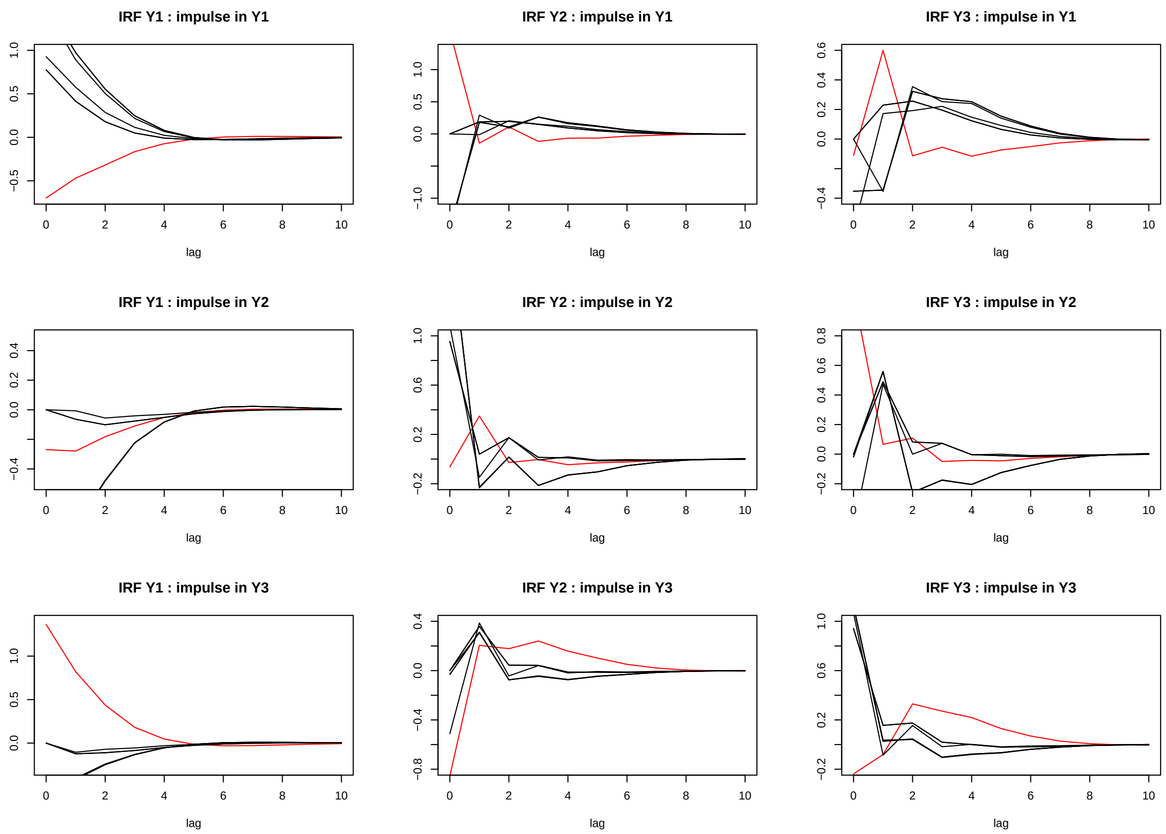

Figure 7: Comparison of impulse responses for different permutations

The means of the impulse response functions calculated via orthogonalization are depicted in black, whereas the theoretical impulse responses are shown in red. As can be seen in figure 7 the estimated impulse response functions can differ wildly from the theoretical responses and even though some permutations lie closely to the correct responses, there is no possibility to decide which ones are close.

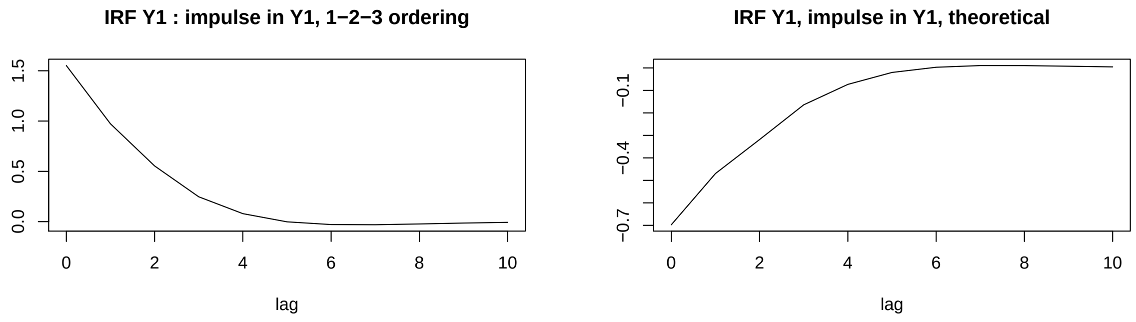

Figure 8: 1-2-3 order irf and Theoretical irf

An example for a particularly strong difference can be found in the comparison between the theoretical impulse response functions for Y_1 to a shock in Y_1 itself compared to the one obtained from the 1-2-3 order shown in figure 8. Even though both functions tend to zero as the lag increases, which is to be expected for a stationary process, the differences are substantial: the theoretical impulse response function is negative for every lag and approximately concave, whereas the estimated function is positive and convex. These differences could be immensely important if one tries to analyse a real economic process.

Since there is no correct order of components it is not possible to create similar confidence intervals as in chapter 3. So it is not possible to look at how often the estimated impulse response functions lie in the respective confidence intervals. Nevertheless the findings up to this point once again support my theoretical claim that one has to be very careful when using Cholesky decomposition to orthogonalize the error terms, since the deviations from the theoretical impulse response functions can be quite extreme, to the point where instead of a positive response the estimate delivers a negative response.

5 Summary

In this paper I addressed the problem of exchanging the components in the reduced form of an SVAR[1] process when analysing the impulse responses using Cholesky decomposition. My concerns about the problem of misspecification in the order of components, which I presented in a theoretical analysis in chapter 2 were confirmed by the simulations conducted in chapter 3 and 4.

For the case that $\mathbf{B_0}$ is a lower triangular matrix, presented in chapter 3, I found that even though some permutations show similar patterns as the correct order, especially if specific permutations are chosen, the differences in general are quite substantial. Since one cannot identify the correct order from the data alone, economic theory has to be used as a foundation to make sure that one chooses the correct permutation, even if the model is recursively identified. Some points on how to find additional restrictions are named in chapter 6.

If there is no reason to assume that $\mathbf{B_0}$ is a lower triangular matrix in any permutation of components, the problems can be even more substantial if one uses orthogonalization to conduct impulse response analysis. Since the method is not suitable for this case, the results can differ wildly from the correct impulse responses, which shows that the method should only be used when the assumption of triangularity is justified theoretically.

In summary my theoretical findings and the results of the simulations conducted in this paper confirm the idea that Impulse Response Analysis via orthogonalization has to be done carefully, as it can lead to incorrect results when used on a misspecified model or in cases where the general methodology is not applicable. These incorrect results can indicate completely different relations between the components of a multivariate time series, that if interpreted could lead to false implications on reality, which in turn could be very problematic if applied to politics.

6 Outlook

In light of the findings of the theoretical analysis and the results from the simulations, the question is how to deal with the problems that have been highlighted in this paper. First, how to get information about $\mathbf{B_0}$? In Kilian and Lütkepohl 2017 the authors explain that from Σ_ϵ one is able to obtain $\frac{K(K+1)}{2}$ identifying restrictions for $\mathbf{B_0}$ due to its symmetrical nature. If one wants to uniquely identify $\mathbf{B_0}$ additional restrictions are necessary, which can come from different sources outside the data. The authors identify multiple possibilities to obtain these in chapter 8.3. In some scenarios one might want to impose the structure implied by a particular model from *economic theory*. Other points include *information delays*, meaning that reaction to data is not possible instantaneously as it is released only infrequently, *physical contraints*, *insitutional knowledge* and *assumptions about market structure*.

Using restrictions from these sources one might also be able to reasonably assume recursive identification in a specific order, which in turn would justify impulse response analysis using Cholesky decomposition.

Another possibility is a different approach to finding the impulse responses that is invariant to a change in the order of components. In Pesaran and Shin 1998, the authors construct generalized impulse response functions which do not rely on orthogonalization. Instead of relying on a Cholesky decomposition the authors obtain the impulse responses by shocking the internally correlated error terms. Using an assumed or observed distribution of the errors, they then integrate out the effects created by correlation between the individual error terms.

In Jordà 2005, the author uses a local projections approach to compute impulse responses. This means, that the methodology does not rely on estimating the underlying process - completely avoiding VAR estimation. The author states that this approach has multiple advantages: *simpler estimation*, *more robust to misspecification*, *easier inference* and *easier accommodation of non-linear or flexible specifications*. Barnichon and Brownlees 2019 further develop this approach using smooth local projections to improve precision, while keeping the advantages described before.

An interesting next step to this paper itself would be to perform a modified Monte Carlo simulation in which many data generating processes are procedurally generated and analysed as a whole, analogously to the analysis conducted in this paper. Doing so could remove any doubt that the results found in the simulations provided in chapter 3 and chapter 4 are special to the specific process chosen.

From a theoretical standpoint it would be interesting to further study the problems described in equations 16 and 17. A theoretical analysis of $\tilde{P}$, the matrix obtained via Cholesky decomposition for an incorrect permutation, would enable us to describe the incorrect theoretical impulse response functions, making it possible to further study the effects of an incorrect order of components.

7 Bibliography

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8 Appendix

On page 7, realisation of the data generating process

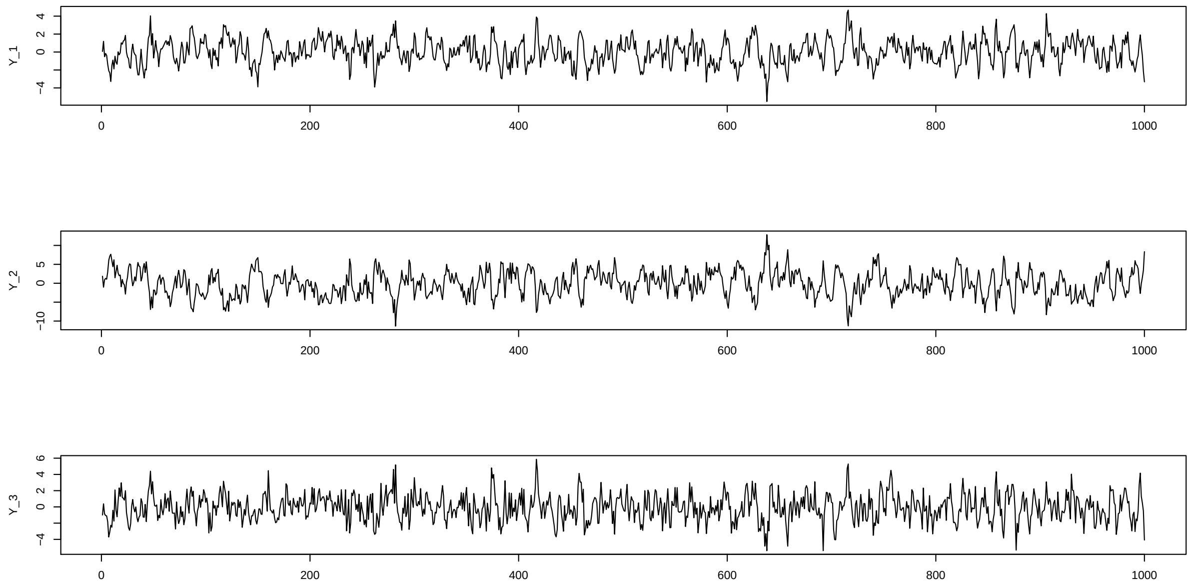

Figure 9: Realisation of the special data generating process

On page 13, realisation of the data generating process

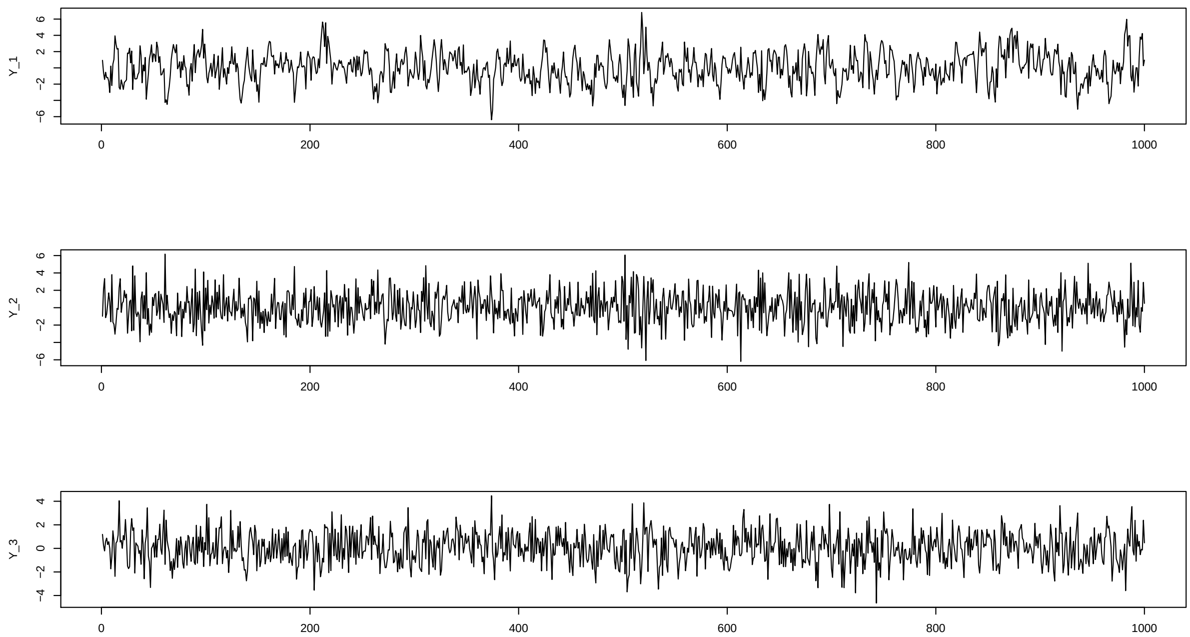

Figure 10: Realisation of the general data generating process

On page 6; proof that $\mathbf{A^{213}B^{213}} = [\mathbf{AB}]^{\mathbf{213}}$

$$
\begin{aligned}
AB &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} \\
A^{213}B^{213} &= \begin{bmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & a_{31} & a_{33} \end{bmatrix} \begin{bmatrix} b_{22} & b_{21} & b_{23} \\ b_{12} & b_{11} & b_{13} \\ b_{32} & b_{31} & b_{33} \end{bmatrix} \\
&= \begin{bmatrix} a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} \\
&= [AB]^{213}
\end{aligned}
\tag{25}
$$

It follows that

$$
\mathbf{A^{213}(A^{-1})^{213}} = \mathbf{1_3}^{\mathbf{213}} = \mathbf{1_3} \iff (\mathbf{A^{-1}})^{\mathbf{213}} = (\mathbf{A^{213}})^{\mathbf{-1}} \tag{26}
$$

Ich versichere hiermit, dass ich die vorstehende Hausarbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe, dass die vorgelegte Arbeit noch an keiner anderen Hochschule zur Prüfung vorgelegt wurde und dass sie weder ganz noch in Teilen bereits veröffentlicht wurde. Wörtliche Zitate und Stellen, die anderen Werken dem Sinn nach entnommen sind, habe ich in jedem einzelnen Fall kenntlich gemacht.

Jakob Ralph Jürgens