Handout 10 - ECON703 (Fall 2023)

1 Linear Algebra

Definition 1.1 (Identity Matrix). *Let $A \in \mathbb{R}^{n\times n}$ be square matrix. We call A the n-dimensional identity matrix, denoted $\mathbb{I}_n$, if*

$$\forall i = 1,\dots,n: \quad A_{i,i} = 1 \quad \textit{and} \quad \forall i \neq j: \quad A_{i,j} = 0$$

Definition 1.2 (Matrix Addition). *Let A and B be two $m \times n$ matrices. Then, their sum is defined elementwise, i.e.*

$$\forall i = 1,\dots,m \, \forall j = 1,\dots,n: \quad (A+B)_{i,j} = A_{i,j} + B_{i,j}$$

Definition 1.3 (Scalar Multiplication). *Let A be a two $m \times n$ matrix and $\lambda \in \mathbb{R}$ be some scalar. Then, the scalar product of λ with A is defined elementwise, i.e.*

$$\forall i = 1,\dots,m \, \forall j = 1,\dots,n: \quad (\lambda A)_{i,j} = \lambda A_{i,j}$$

Definition 1.4 (Matrix Multiplication). *The product of two matrices A and B, denoted by AB, is defined if and only if the number of columns of A is equal to the number of rows of B. It is* **NOT** *commutative.*

Let $A \in \mathbb{R}^{m\times n}$ and $B = \mathbb{R}^{n\times p}$, then the matrix product AB exists and is defined componentwise by

$$\forall i = 1,\dots,m \, \forall j = 1,\dots,p: \quad (AB)_{i,j} = \sum_{r=1}^{n} A_{i,r} B_{r,j}$$

This product is an $m \times p$ matrix.

There are other types of matrix products, for example, the Hadamard product and Kronecker product. These are used less often, but you will come across them, for example, in Econometrics.

- `https://en.wikipedia.org/wiki/Hadamard_product_(matrices)`
- `https://en.wikipedia.org/wiki/Kronecker_product`

Definition 1.5 (Transposition of a Matrix). *The transposition of a matrix $A \in \mathbb{R}^{m\times n}$, denoted $A' \in \mathbb{R}^{n\times m}$ is defined element-wise as*

$$\forall i = 1,\dots,m \, \forall j = 1,\dots,n: \quad (A')_{i,j} = A_{j,i}$$

and is a $n \times m$ matrix.

Definition 1.6 (Determinant of a Matrix). *Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. Its determinant is defined via the Leibniz formula.*

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

https://en.wikipedia.org/wiki/Leibniz_formula_for_determinants

In your first year, you will rarely (if at all) need to use determinants for matrices larger than 3×3. So, let's give the formulas for 2×2 and 3×3 matrices more directly.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - cb$$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = aei + bfg + cdh - ceg - bdi - afh$$

Definition 1.7 (Matrix Inverse)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. If there exists some $B \in \mathbb{R}^{n\times n}$ such that $AB = \mathbb{I}_n$, then we call B the matrix inverse of A and denote it by A^{-1}. It exists if $det(A) \neq 0$.*

Definition 1.8 (Symmetric Matrix)**.** *We say a square matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if*

$$\forall i = 1, \dots, n \, \forall j = 1, \dots, n \quad A_{i,j} = A_{j,i}$$

This is equivalent to saying that $A = A'$.

Definition 1.9 (Positive / Negative (Semi-)Definite Matrix)**.** *We say a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive (negative) definite if*

$$\forall x \in \mathbb{R}^n \backslash \{0\} : \quad x'Ax > (<) 0$$

We call it positive (negative) semidefinite if

$$\forall x \in \mathbb{R}^n \backslash \{0\} : \quad x'Ax \geq (\leq) 0$$

Definition 1.10 (Trace)**.** *The trace of a square matrix $A \in \mathbb{R}^{n\times n}$, denoted $tr(A)$, is the sum of its entries on the main diagonal.*

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Definition 1.11 (Eigenvectors and Eigenvalues)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. Eigenvectors of A are vectors $\mu \in \mathbb{R}^n$ that fulfill the following property:*

$$\exists \lambda \in \mathbb{R} \quad s.t. \quad A\mu = \lambda\mu$$

The λ corresponding to some Eigenvector μ is called the corresponding Eigenvalue. We typically consider Eigenvectors whose length is normalized to one.

2 Linearity

Definition 2.1 (Linear Transformation)**.** *Let W and V be vector spaces over $\mathbb{R}$ and let $f : V \to W$ be a function between them. We say that f is a linear transformation or linear map, if*

$$\forall u, v \in V : \quad f(u + v) = f(u) + f(v)$$
$$\forall v \in V \, \forall \lambda \in \mathbb{R} : \quad f(\lambda v) = \lambda f(v)$$

Example 2.1 (Matrix)**.** *Let $A \in \mathbb{R}^{m\times n}$ be a matrix and define the function $f : \mathbb{R}^n \to \mathbb{R}^m$ by*

$$f(x) = Ax$$

then f is a linear transformation.

Proof. Check the two properties with the definition of matrix multiplication given above. □

3 Integration

The two integration rules we use most often during the first year are integration by parts and integration by substitution.

Theorem 3.1 (Integration by Parts)**.**

$$\int_a^b u(x)v'(x)\mathrm{d}x = [u(x)v(x)]\Big|_a^b - \int_a^b u'(x)v(x)\mathrm{d}x$$

Example 3.1 (Integration by Parts)**.** *Consider the following integral.*

$$\int_0^2 xe^{2x}\mathrm{d}x$$

Then, in our notation from before, we can set

$$u(x) = x \implies u'(x) = 1$$
$$v'(x) = e^{2x} \implies v(x) = \frac{1}{2}e^{2x}$$

Using partial integration, this gives us the following

$$\int_0^2 xe^{2x}\mathrm{d}x = \left[\frac{x}{2}e^{2x}\right]\Big|_0^2 - \int_0^2 \frac{1}{2}e^{2x}\mathrm{d}x = e^4 - \left(\frac{1}{4}e^4 - \frac{1}{4}\right) = \frac{1}{4} + \frac{3}{4}e^4 \approx 41.199$$

Theorem 3.2 (Integration by Substitution)**.** *Let $g : [a, b] \to \mathbb{R}$ be a continuously differentiable function and $f : \mathbb{R} \to \mathbb{R}$ a continuous function. Then*

$$\int_a^b f(g(x))g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u$$

Example 3.2 (Integration by Substitution)**.** *Consider the following integral.*

$$\int_0^{\frac{\pi}{2}} (\sin(x))^2 \cos(x)\mathrm{d}x$$

and set $g(x) = \sin(x)$. Then, by integration by substitution:

$$\int_0^{\frac{\pi}{2}} (\sin(x))^2 \cos(x)\mathrm{d}x = \int_{sin(0)}^{sin(\frac{\pi}{2})} u^2\mathrm{d}u = \int_0^1 u^2\mathrm{d}u = \left[\frac{1}{3}u^3\right]\Big|_0^1 = \frac{1}{3}$$

4 Taylor Approximation

Theorem 4.1 (Taylor Series (Univariate))**.** *Let $f \in \mathcal{C}^k(A, \mathbb{R})$ for some set $A \subset \mathbb{R}$. Let $x_0 \in int(A)$. Assume that f is k times differentiable at x_0. Then, there exists $h_k(x)$ with $\lim_{x\to x_0} h_k(x) = 0$ such that*

$$f(x) = \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(x_0)(x - x_0)^i + h_k(x)(x - x_0)^k$$

We call the following term the k'th order Taylor polynomial:

$$P_k(x) = \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(x_0)(x - x_0)^i$$

and the other term the remainder term: $R_k(x) = h_k(x)(x - x_0)^k$.

We can give explicit formulas for the remainder term that can help us gauge our approximation's accuracy.

Theorem 4.2 (Lagrange-Form and Cauchy-Form of the Remainder)**.** *Let $f \in \mathcal{C}^k(A, \mathbb{R})$ for some set $A \subset \mathbb{R}$. Let $x_0 \in int(A)$. Assume that f is $k + 1$ times differentiable on the interval (x, x_0) with $f^{(k)}$ being continuous on $[x, x_0]$. Then, we can find an explicit formula for the remainder term of the following form called the Lagrange form:*

$$\exists \xi_L \in [x, x_0] \quad s.t. \quad R_k(x) = \frac{f^{(k+1)}(\xi_L)}{(k + 1)!}(x - x_0)^{k+1}$$

Similarly, we can find an expression of the following form, called the Cauchy form:

$$\exists \xi_C \in [x, x_0] \quad s.t. \quad R_k(x) = \frac{f^{(k+1)}(\xi_C)}{k!}(x - \xi_C)^k(x - x_0)$$

Example 4.1 (First Order Logarithm Approximation)**.** *Let $f(x) = \log(1 + x)$. Then we can use the first Taylor polynomial around $x = 0$ to approximate the logarithm for small values of x as*

$$\log(1 + x) \approx \log(1) + x\frac{\partial}{\partial x}\log(1 + x)\Big|_{x=0} = 0 + \frac{x}{1 + 0} = x$$