Handout 2 - ECON703 (Fall 2023)

1 Supremum and Infimum

Example 1.1 (Approaching One)**.** *Let* $X = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$*. Then* $\sup X = 1$.

Proof. **Step 1:** Show that 1 is an upper bound of X.
Observe that $\forall n \in \mathbb{N}: \frac{1}{n} > 0$. Thus $\forall n \in \mathbb{N}: 1 - \frac{1}{n} \leq 1$.

Step 2: Show that for any upper bound u of X, we have $u \geq 1$.
Let u be an upper bound of X and assume (for the sake of contradiction) that $u < 1$. But then:

$$\exists k \in \mathbb{N} \quad \text{s.t.} \quad \frac{1}{k} < 1 - u \implies \exists k \in \mathbb{N} \quad \text{s.t.} \quad 1 - \frac{1}{k} > u \implies \exists x \in X \quad \text{s.t.} \quad x > u$$

Contradiction! Thus, 1 is the supremum of X. □

Example 1.2 (Supremum and Infimum of a Square)**.** *Let* $A = (0,1) \times (0,1) \subset \mathbb{R}^2$ *and consider the following partial order* $\preceq$ *on* $\mathbb{R}^2$*:*

$$\forall a, b \in A: \ (a \preceq b) \iff (b_1 \geq a_1 \wedge b_2 \geq a_2)$$

then

$$\sup A = (1,1) \quad \textit{and} \quad \inf A = (0,0)$$

Proof. Let's only consider the supremum; the infimum works analogously.
Step 1: Show that $(1,1)$ is an upper bound of A with respect to the partial order.

$$\forall a \in A: \ a_1 < 1 \wedge a_2 < 1 \implies \forall a \in A: \ a \preceq (1,1)$$

Therefore, $(1,1)$ is an upper bound of A with respect to the partial order $\preceq$.

Step 2: Show that for any other upper bound u of A, we have $(1,1) \preceq u$.
Let u be an upper bound of A. Then:

$$\begin{aligned} \forall a \in A: \ a \preceq u &\implies \forall a \in A: \ u_1 \geq a_1 \wedge u_2 \geq a_2 \\ &\implies \forall x \in (0,1): \ u_1 \geq x \wedge u_2 \geq x \\ &\implies u_1 \geq 1 \wedge u_2 \geq 1 \\ &\implies (1,1) \preceq u \end{aligned}$$

Thus, $(1,1)$ is the supremum of A. □

2 Metric Spaces and Open/Closed Sets

Example 2.1 (Euclidean Space with Maximum Distance)**.** *Let* $n \in \mathbb{N}$*. Then* $\mathbb{R}^n$ *combined with the following operations is a vector space over the real numbers* $\mathbb{R}$*.*

$$+_{\mathbb{V}}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \quad \forall x, y \in \mathbb{R}^n \quad (x +_{\mathbb{V}} y)_i = x_i + y_i \quad \forall i = 1, \ldots, n$$

$$\cdot_{\mathbb{V}}: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \quad \forall x \in \mathbb{R}^n \, \forall \lambda \in \mathbb{R} \quad (\lambda \cdot_{\mathbb{V}} x)_i = \lambda x_i \quad \forall i = 1, \ldots, n$$

We call it the n-dimensional Euclidean Space. The maximum distance on this space is

$$d(x,y) = \max_{i=1,\ldots,n} |x_i - y_i|$$

Together, these form a metric space.

Proof. To show that this is a metric space, we have to show that d fulfills the properties of a metric.

1. $\forall v \in \mathbb{V}: \ d(v,v) = 0$
2. **Positivity**: $\forall u, v \in \mathbb{V}$ with $u \neq v: \ d(u,v) > 0$
3. **Symmetry**: $\forall u, v \in \mathbb{V}: \ d(u,v) = d(v,u)$
4. **Triangle Inequality**: $\forall u, v, w \in \mathbb{V}: \ d(u,w) \leq d(u,v) + d(v,w)$

Let's check them one by one.

1. Holds since $\forall v \in \mathbb{R}^n \ \forall i = 1, \ldots, n: \ v_i - v_i = 0 \quad \checkmark$
2. Holds since $\forall a \in \mathbb{R}: \ |a| \geq 0 \quad \checkmark$
3. Holds since $\forall a, b \in \mathbb{R}: \ |a - b| = |b - a| \quad \checkmark$
4. Let's check this one in a bit more detail. Let $x, y, z \in \mathbb{R}^n$

$$\begin{aligned} d(x,z) = \max_{i=1,\ldots,n} |x_i - z_i| &= \max_{i=1,\ldots,n} |x_i - y_i + y_i - z_i| \\ &\leq \max_{i=1,\ldots,n} \left[\, |x_i - y_i| + |y_i - z_i| \,\right] \\ &\leq \max_{i=1,\ldots,n} |x_i - y_i| + \max_{j=1,\ldots,n} |y_j - z_j| = d(x,y) + d(y,z) \quad \checkmark \end{aligned}$$

□

One important thing to understand about open and closed sets is the following:

> A subset is not like a door: it can be open, closed, both, or neither.

Example 2.2 (Open and Closed Sets)**.** *Consider the following Sets as subsets of $\mathbb{R}^2$ and let $d(x,y)$ be the Euclidean distance between two points $x, y \in \mathbb{R}^2$.*

1. $\mathbb{R}^2$
2. $\emptyset$
3. $A = \{x \in \mathbb{R}^2 \mid d(x,(0,0)) \leq 1\}$
4. $B = \{x \in \mathbb{R}^2 \mid d(x,(0,0)) < 1\}$
5. $C = \{x \in \mathbb{R}^2 \mid d(x,(0,0)) = 1\}$
6. $D = A \backslash \{x \in \mathbb{R}^2 \mid x_1 > 0\}$
7. $E = B \backslash \{x \in \mathbb{R}^2 \mid x_1 > 0\}$
8. $F = A \backslash \{x \in \mathbb{R}^2 \mid x_1 \geq 0\}$
9. $G = B \backslash \{x \in \mathbb{R}^2 \mid x_1 \geq 0\}$
10. $H = \{(0,0)\}$

Are they open, closed, both, or neither?

3 Sequences and Series

Definition 3.1 (Convergent Sequence)**.** *Let (X, d_X) be a metric space and $(x_n)_{n \in \mathbb{N}}$ be a sequence in X. We say that x_n converges to $\overline{x}$, denoted by $\lim_{n \to \infty} x_n = \overline{x}$ or $x_n \to \overline{x}$ as $n \to \infty$, if*

$$\forall \epsilon > 0 \ \exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n > N_\epsilon: \ d_X(x_n, \overline{x}) < \epsilon$$

Example 3.1 (Geometric Series)**.** *Let $(x_n)_{n \in \mathbb{N}}$ be given by $x_n = \sum_{i=0}^{n-1} r^i$ for $r \in (0,1)$. Then $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence with $\lim_{n \to \infty} x_n = \frac{1}{1-r}$.*

Proof. First, we show that the following formula gives the value of a partial sum of the geometric series.

$$x_n = \sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$$

We will do so using a proof by induction.

- **Base Case**: $n = 1$

$$x_1 = \sum_{i=0}^{0} r^i = r^0 = 1 = \frac{1-r^1}{1-r}$$

- **Induction Step**: Assume that $x_n = \frac{1-r^n}{1-r}$. Then

$$x_{n+1} = x_n + r^n = \frac{1-r^n}{1-r} + r^n = \frac{(1-r^n)+(1-r)r^n}{1-r} = \frac{(1-r^n)+(r^n-r^{n+1})}{1-r} = \frac{1-r^{n+1}}{1-r}$$

  Thus, we have shown that the assumed formula for the partial sums holds.

If the limit exists, this gives us the following statement.

$$\sum_{i=1}^{\infty} r^i = \lim_{n\to\infty} \sum_{i=1}^{n} r^i = \lim_{n\to\infty} \frac{1-r^n}{1-r}$$

We can show that the second limit exists in the following way.

$$\lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r}\left(1 - \lim_{n\to\infty} r^n\right) = \frac{1}{1-r}(1-0) = \frac{1}{1-r}$$

where the second equality holds since $|r| < 1$. Thus, the series is convergent and converges to $\frac{1}{1-r}$. □

Example 3.2 (Harmonic Series). *Let $(x_n)_{n\in\mathbb{N}}$ be given by $x_n = \sum_{i=1}^{n} \frac{1}{i}$. Then $(x_n)_{n\in\mathbb{N}}$ diverges.*

Proof. Observe that we can group the terms of the harmonic series as follows:

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{2^{i-1}} \frac{1}{2^{i-1}+j}\right) = 1 + \frac{1}{2} + (\frac{1}{3}+\frac{1}{4}) + (\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}) + \dots$$

But then

$$\forall i \in \mathbb{N}: \quad \sum_{j=1}^{2^{i-1}} \frac{1}{2^{i-1}+j} \geq \frac{1}{2}$$

and thus

$$\sum_{i=1}^{\infty} \frac{1}{i} \geq \sum_{i=1}^{\infty} \frac{1}{2} = \infty$$

Thus $\sum_{i=1}^{\infty} \frac{1}{i}$ clearly diverges. □

4 Continuity

Definition 4.1 (Continuity). *Let (X, d_X) and (Y, d_Y) be a metric spaces and $f : X \to Y$ a function from X to Y. We say that f is continuous at a point $x \in X$ if*

$$\forall \epsilon > 0\, \exists \delta > 0 \quad \textit{such that} \quad \forall x' \in X: \quad d_X(x,x') < \delta \implies d_Y(f(x), f(x')) < \epsilon$$