Handout 5 - ECON703 (Fall 2023)

1 Vector Spaces and Norms

In the following, whenever we use the term field ($\mathbb{F}$), you can think of $\mathbb{R}$ together with our usual addition and multiplication. We can define many of these objects more generally, but let's stay in the reals for now.

Definition 1.1 (Vector Space)**.** *A vector space* $(\mathbb{V}, +_{\mathbb{V}}, \cdot_{\mathbb{V}})$ *over a field* $\mathbb{F}$ *(with addition* $+_{\mathbb{F}}$ *and multiplication* $\cdot_{\mathbb{F}}$*) is a non-empty set* $\mathbb{V}$ *together with two binary operations* $+_{\mathbb{V}} : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ *and* $\cdot_{\mathbb{V}} : \mathbb{F} \times \mathbb{V} \to \mathbb{V}$ *such that the following properties hold.*

- *Associativity of Vector Addition:*
$$\forall u, v, w \in \mathbb{V} : \ u + (v + w) = (u + v) + w$$
- *Commutativity of Vector Addition:*
$$\forall u, v \in \mathbb{V} : \ u + v = v + u$$
- *Existence of an Identity Element of Vector Addition:*
$$\exists 0_V \in \mathbb{V} \ s.t. \ \forall v \in \mathbb{V} : \ v + 0_V = v$$
- *Existence of Inverse Elements of Vector Addition:*
$$\forall v \in \mathbb{V} \ \exists (-v) \in \mathbb{V} \ s.t. \ v + (-v) = 0_V$$
- *Compatibility of Scalar Multiplication with Field Multiplication:*
$$\forall v \in \mathbb{V} \ \forall \lambda, \mu \in \mathbb{F} : \ \lambda \cdot_{\mathbb{V}} (\mu \cdot_{\mathbb{V}} v) = (\lambda \cdot_{\mathbb{F}} \mu) \cdot_{\mathbb{V}} v$$
- *Existence of an Identity Element of Scalar Multiplication:*
$$\forall v \in \mathbb{V} : \ 1_F \cdot_{\mathbb{V}} v = v$$
- *Distributivity of Scalar Multiplication w.r.t. Vector Addition:*
$$\forall u, v \in \mathbb{V} \ \forall \lambda \in \mathbb{F} : \ \lambda \cdot_{\mathbb{V}} (u +_{\mathbb{V}} v) = \lambda \cdot_{\mathbb{V}} v +_{\mathbb{V}} \lambda \cdot_{\mathbb{V}} u$$
- *Distributivity of Scalar Multiplication w.r.t. Field Addition:*
$$\forall v \in \mathbb{V} \ \forall \lambda, \mu \in \mathbb{F} : \ (\lambda +_{\mathbb{F}} \mu) \cdot_{\mathbb{V}} v = \lambda \cdot_{\mathbb{V}} v +_{\mathbb{V}} \mu \cdot_{\mathbb{V}} v$$

Example 1.1 (Two-Dimensional Real Vectors)**.** *The set of all ordered pairs of real numbers, i.e.,* $\mathbb{R}^2$ *together with component-wise addition and scalar multiplication, is a vector space over* $\mathbb{R}$*.*

$$+ : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \quad \textit{defined by} \quad (x, y) + (a, b) = (x + a, y + b)$$
$$\cdot : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \quad \textit{defined by} \quad \lambda \cdot (x, y) = (\lambda x, \lambda y)$$

Proof. Check the properties... □

Definition 1.2 (Norm). *A function $||\cdot||:\mathbb{V}\to\mathbb{R}$ from a vector space $\mathbb{V}$ to the underlying field $\mathbb{R}$ is called a norm if it fulfills the following properties.*

1. **Triangle Inequality**:
$$\forall u,v\in\mathbb{V}: \quad ||u+v|| \le ||u|| + ||v||$$
2. **Absolute Homogeneity**:
$$\forall v\in\mathbb{V},\lambda\in\mathbb{F}: \quad ||\lambda v|| = |\lambda|\,||v||$$
3. **Positive Definiteness**:
$$||v|| = 0_{\mathbb{F}} \implies v = 0_{\mathbb{V}}$$

A function that fulfills properties 1 and 2 is called a seminorm.

Example 1.2 (ℓ_p-Norm). *For $p\in[1,\infty)$, the following formula defines the ℓ_p-norm.*

$$||x||_p = \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}$$

This is a norm on an n-dimensional Euclidean space.

Theorem 1.1 (Distance induced by a Norm). *A norm $||\cdot||:\mathbb{V}\to\mathbb{R}$ on a vector space $(\mathbb{V},+_{\mathbb{V}},\cdot_{\mathbb{V}})$ induces a distance on $(\mathbb{V},+_{\mathbb{V}},\cdot_{\mathbb{V}})$ as follows:*

$$d(x,y) = ||x-y||$$

Proof. Check the properties of a distance:

- $\forall v\in\mathbb{V}:\ d(v,v)=0$ Holds by positive definiteness of the underlying norm ✓
- **Positivity**: Holds by positivity of the underlying norm ✓
- **Symmetry**: Holds by commutativity of vector space addition ✓
- **Triangle Inequality**: Let $x,y,z\in\mathbb{V}$
$$d(x,y) = ||y-x|| = ||y-z+z-x|| \le ||y-z|| + ||z-x|| = d(y,z) + d(z,x) \quad ✓$$

□

2 Inner Products and the Cauchy-Schwarz Inequality

Definition 2.1 (Inner Product). *An inner product on a vector space $\mathbb{V}$ over $\mathbb{R}$ is a function $\langle\cdot,\cdot\rangle:\mathbb{V}\times\mathbb{V}\to\mathbb{R}$ satisfying the following properties.*

- **Symmetry**: $\forall v,u\in\mathbb{V}:\langle v,u\rangle = \langle u,v\rangle$
- **Linearity in the first Argument**: $\forall u,v,w\in\mathbb{V}\,\forall\lambda,\mu\in\mathbb{F}:\ \langle\lambda v+\mu u,w\rangle = \lambda\langle u,w\rangle + \mu\langle v,w\rangle$
- **Positive Definiteness**: $\forall v\in\mathbb{V}:\ (v\ne 0) \implies (\langle v,v\rangle > 0)$

Example 2.1 (Canonical Inner Product on Euclidean Space)**.** *The canonical inner product on* $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, *is given by*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

Proof. Check the properties of the inner product definition.

- **Conjugate Symmetry**: Holds ✓
- **Linearity in the first Argument**: Holds ✓
- **Positive Definiteness**: Holds ✓

□

Definition 2.2 (Inner Product Space)**.** *An inner product space is a vector space* $\mathbb{V}$ *over some field* $\mathbb{F}$ *together with an inner product* $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{F}$.

Theorem 2.1 (Cauchy-Schwarz Inequality)**.** *Let* $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ *be an inner product space. Then*

$$\forall u, v \in \mathbb{V} : \; |\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle$$

In Euclidean space with the standard inner product, the Cauchy-Schwarz inequality becomes:

$$\left(\sum_{i=1}^{n} u_i v_i \right)^2 \leq \left(\sum_{i=1}^{n} u_i^2 \right) \left(\sum_{i=1}^{n} v_i^2 \right)$$

with equality if and only if u *and* v *are linearly dependent.*

Proof. Due to time constraints: see `https://en.wikipedia.org/wiki/Cauchy-Schwarz_inequality#Proofs` □

3 The Space of Continuous and Bounded Functions

Definition 3.1 (Uniformly Bounded Set of Functions)**.** *Let* X *be an arbitrary set and* $\mathcal{F} = \{f_i : X \to \mathbb{R} \mid i \in \mathcal{I}\}$ *be a set of functions indexed by some index set* $\mathcal{I}$. *We say that* $\mathcal{F}$ *is bounded uniformly if*

$$\exists M \in \mathbb{R} \quad \textit{such that} \quad \forall i \in \mathcal{I} \, \forall x \in X : \; |f_i(x)| \leq M$$

Definition 3.2 (Space of Continuous and Bounded Functions)**.** *The space of continuous and bounded functions from a set* $X \subset \mathbb{R}^n$ *to* $\mathbb{R}$, *denoted by* $\mathcal{C}_b(X)$, *is defined as the vector space consisting of the set*

$$\{f : X \to \mathbb{R} \mid f \, \textit{continuous and bounded}\}$$

together with pointwise addition and scalar multiplication.

$$\begin{aligned} + : \mathcal{C}_b(X) \times \mathcal{C}_b(X) \to \mathcal{C}_b(X) \quad &\textit{given by} \qquad \forall f, g \in \mathcal{C}_b(X) \, \forall x \in X : \; (f+g)(x) = f(x) + g(x) \\ \cdot : \mathbb{R} \times \mathcal{C}_b(X) \to \mathcal{C}_b(X) \quad &\textit{given by} \qquad \forall f \in \mathcal{C}_b(X) \, \forall \lambda \in \mathbb{R} \, \forall x \in X : \; (\lambda f)(x) = \lambda f(x) \end{aligned}$$

One important thing to note is that not all of these functions have to be bounded by the same bound M. In fact, if we assume that they are, this is not a vector space.

Theorem 3.1 (The space of Continuous and Bounded Functions with the sup-norm is a Complete Metric Space)**.** $\mathcal{C}_b(X)$ *together with the operations defined above and the distance induced by the* sup*-norm*

$$d(f, g) = \sup_{x \in X} |g(x) - f(x)|$$

is a complete metric space.

Proof. Let $f_i : X \to \mathbb{R}$ be a Cauchy sequence with respect to the sup-distance, i.e.

$$\forall \epsilon > 0 \, \exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n, m > N_\epsilon : \sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon$$

Then $\forall x \in X$ the sequence $(f_i(x))_{i \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and thus convergent. That means that the pointwise limit of our function sequence exists. Let $f : X \to \mathbb{R}$ be the pointwise limit of f_i, i.e.

$$\forall x \in X : \quad f(x) = \lim_{i \to \infty} f_i(x)$$

This will be our candidate for the limit of our function sequence in the sup-distance.

Uniform Convergence:

$$\forall \epsilon > 0 \, \exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n, m > N_\epsilon : \sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon$$

and thus, by linearity of the absolute value distance, we have

$$\forall x \in X : \, |f(x) - f_m(x)| = |\lim_{i \to \infty} f_i(x) - f_m(x)| = \lim_{i \to \infty} |f_i(x) - f_m(x)| \leq \epsilon$$

But then

$$\forall m > N_\epsilon : \, \sup_{x \in X} |f(x) - f_m(x)| \leq \epsilon$$

and thus, we have uniform convergence.

Boundedness of f:
Set $\epsilon = \frac{1}{2}$ and choose $N \in \mathbb{N}$ such that

$$\forall m, n \geq N : \, \sup_{x \in X} |f_n(x) - f_m(x)| < \frac{1}{2}$$

Then using the argument used for uniform convergence, we get

$$\begin{aligned} \forall x \in X : \, |f(x)| &\leq |f(x) - f_N(x)| + |f_N(x)| \leq \sup_{x \in X} |f(x) - f_N(x)| + \sup_{x \in X} |f_N(x)| \\ &\leq \frac{1}{2} + \sup_{x \in X} |f_N(x)| \end{aligned}$$

And thus, f is bounded.

Continuity of f:
Choose $\epsilon > 0$ arbitrarily and let $a \in X$ be some point in our domain. Then by uniform convergence to f we have

$$\exists N_\epsilon \in \mathbb{N} \quad \text{such that} \quad \forall n > N_\epsilon : \sup_{x \in X} |f_n(x) - f(x)| < \frac{\epsilon}{3}$$

Since $\forall n \in \mathbb{N}$ f_n is continuous at a we have

$$\exists \delta > 0 \quad \text{such that} \quad |x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\epsilon}{3}$$

Combining these, we can obtain the following for sufficiently large values of n:

$$\forall x \in X : |x - a| < \delta \implies |f(x) - f(a)| \leq |f_n(x) - f(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < \epsilon$$

and thus f is continuous at a. Since a was chosen arbitrarily, f is continuous. $\square$