Handout 7 - ECON703 (Fall 2023)

1 Graph of a Function

Definition 1.1 (Graph of a Function). *The graph of a function $f : X \to Y$ is defined as the set of ordered pairs $\Gamma(f) = \{(x, y) \mid f(x) = y\} \subset X \times Y$.*

Definition 1.2 (Hypograph and Epigraph). *The Epigraph of a function $f : X \to \mathbb{R}$ is defined as*

$$Epi(f) = \{(x, y) \mid y \geq f(x)\} \subset X \times \mathbb{R}$$

Its Hypograph is defined as

$$Hypo(f) = \{(x, y) \mid y \leq f(x)\} \subset X \times \mathbb{R}$$

We can extend those definitions to functions whose codomain is a partially ordered set.

2 Isoquants and Contour Sets

Definition 2.1 (Isoquant). *Let $f : X \to \mathbb{R}$ be a function from some set X to $\mathbb{R}$. Then the Isoquant of f for the value $z \in \mathbb{R}$ is defined as*

$$\{x \in X \mid f(x) = z\}$$

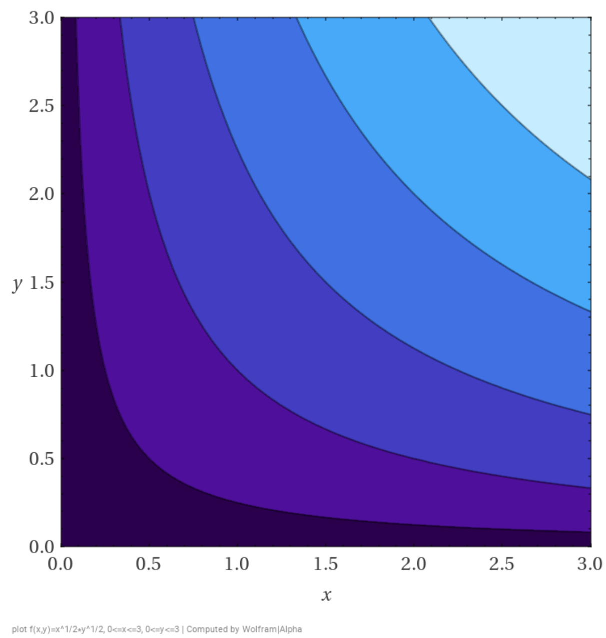

Figure 1: A contour plot of a Cobb-Douglas Function

Definition 2.2 (Upper Contour Set). *Let $f : X \to \mathbb{R}$ be a function from some set X to $\mathbb{R}$. Then the upper contour set of f at $z \in \mathbb{R}$ is defined as*

$$\mathcal{C}_z^+ = \{x \in X \mid f(x) \geq z\}$$

Definition 2.3 (Lower Contour Set). *Let $f : X \to \mathbb{R}$ be a function from some set X to $\mathbb{R}$. Then the lower contour set of f at $z \in \mathbb{R}$ is defined as*

$$\mathcal{C}_z^- = \{x \in X \mid f(x) \leq z\}$$

Like before, we can extend these definitions to functions whose codomain is a partially ordered set.

3 Differentiability

Definition 3.1 (Directional Limits). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function from $\mathbb{R}$ to $\mathbb{R}$ and $I \subset \mathbb{R}$ be an interval. Let $a \in I$. We call $L \in \mathbb{R}$ the left limit of f at a if*

$$\lim_{x \to a^-} = L \iff \forall \epsilon > 0 \, \exists \delta > 0 \quad \text{such that} \quad \forall x \in I : \; (0 < a - x < \delta) \implies (|f(x) - L| < \epsilon)$$

We call $R \in \mathbb{R}$ the right limit of f at a if

$$\lim_{x \to a^+} = R \iff \forall \epsilon > 0 \, \exists \delta > 0 \quad \text{such that} \quad \forall x \in I : \; (0 < x - a < \delta) \implies (|f(x) - R| < \epsilon)$$

Definition 3.2 (Differentiability in one Dimension). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is differentiable at $a \in \mathbb{R}$ if its derivative*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

Definition 3.3 (Differentiable Function in one Dimension). *Let $U \subset \mathbb{R}$ be a subset of the real line. We say that $f : U \to \mathbb{R}$ is a differentiable function if it is differentiable at all $x \in U$.*

Example 3.1 (Differentiable Function). *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then f is a differentiable function.*

Proof.

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$$

Thus, the limit exists, and we even found the derivative. $\square$

Example 3.2 (Non-Differentiable Function). *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} 2x & \text{if} \quad x \leq 2 \\ 3x - 2 & \text{if} \quad x > 2 \end{cases}$$

Then f is not differentiable at $x = 2$.

Proof. Let's check sequences that approach 2 from the left and the right. In this case, these correspond to the left and right limits at $x = 2$.

Let $(h_n)_{n \in \mathbb{N}}$ be given by $h_n = 2^{-n}$.

$$\lim_{n \to \infty} \frac{f(2 + h_n) - f(2)}{h_n} = \lim_{n \to \infty} \frac{3(2 + h_n) - 2 - 4}{h_n} = \lim_{n \to \infty} \frac{3h_n}{h_n} = 3$$
$$\lim_{n \to \infty} \frac{f(2 - h_n) - f(2)}{h_n} = \lim_{n \to \infty} \frac{2(2 + h_n) - 4}{h_n} = \lim_{n \to \infty} \frac{2h_n}{h_n} = 2$$

Thus, we have shown that there are sequences converging to 2 such that the function evaluated at these sequences converges to different values. Therefore, the limit, i.e., the derivative, does not exist at $x = 2$. $\square$

4 Cardinal vs. Ordinal Properties

Definition 4.1 (Ordinal Property). *We say that a property of a function is ordinal if it is conserved under any strictly increasing transformation.*

Example 4.1 (Quasiconcavity). *Quasiconcavity is preserved under all strictly increasing transformations, as the upper and lower contour sets of a function are not changed under strictly monotonic transformations.*

Definition 4.2 (Cardinal Property). *A cardinal property of a function is a property that is not necessarily conserved under strictly increasing transformations.*

Example 4.2 (Concavity). *Consider the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $f(x) = \sqrt{x}$. Then f is concave. Consider the transformation $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $g(x) = x^4$, which is clearly strictly increasing. Then $(g \circ f)(x) = x^2$ is a strictly increasing transformation of f that is not concave.*

5 Homogeneous and Homothetic Functions

Definition 5.1 (Linear Cone). *Let $V \subset \mathbb{R}^n$ be a subset of Euclidean space. We call V a linear cone if*

$$\forall x \in V \, \forall \lambda \in \mathbb{R}_{\neq 0} : \quad \lambda x \in V$$

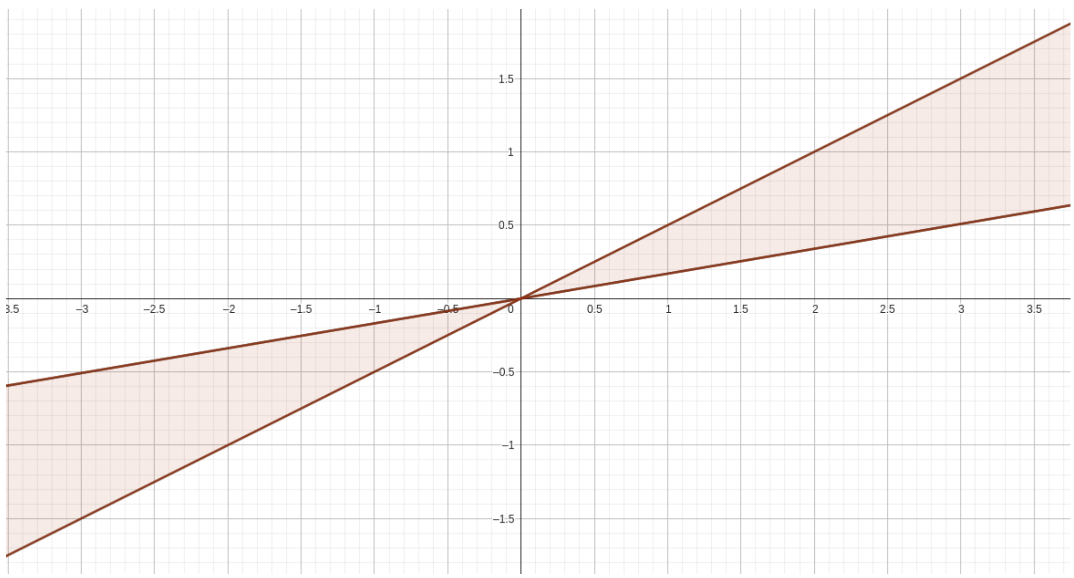

Figure 2: Example of a linear cone in $\mathbb{R}^2$

Definition 5.2 (Homogeneous Function). *Let $V \subset \mathbb{R}^n$ be a linear cone and $f : V \to \mathbb{R}$ be a function from Euclidean space to the real numbers. We say that f is a homogeneous function of degree $k \in \mathbb{N}$ if*

$$\forall \lambda \in \mathbb{R}_{\neq 0} \, \forall x \in V : \quad f(\lambda x) = \lambda^k f(x)$$

Definition 5.3 (Homothetic Function). *We say a function f is homothetic if it is a monotonic transformation of a homogeneous function.*

$\Longrightarrow$ A Homothetic Function has "parallel" isoquants.