Handout 8 - ECON703 (Fall 2023)

1 Monotone Comparative Statics

For more information on monotone comparative statics, see the following paper:

> Milgrom, Paul, and Chris Shannon. "Monotone Comparative Statics."
> Econometrica, Vol. 62, no. 1 (1994): 157–80. `https://doi.org/10.2307/2951479`

Definition 1.1 (Meet and Join)**.** *Let $(X, \preceq)$ be partially ordered set. Then we define the meet and join of two elements $a, b \in X$ by*

$$\begin{aligned} \textit{Join}: &\quad a \vee b = \sup\{a, b\} \\ \textit{Meet}: &\quad a \wedge b = \inf\{a, b\} \end{aligned}$$

if the supremum and infimum exist.

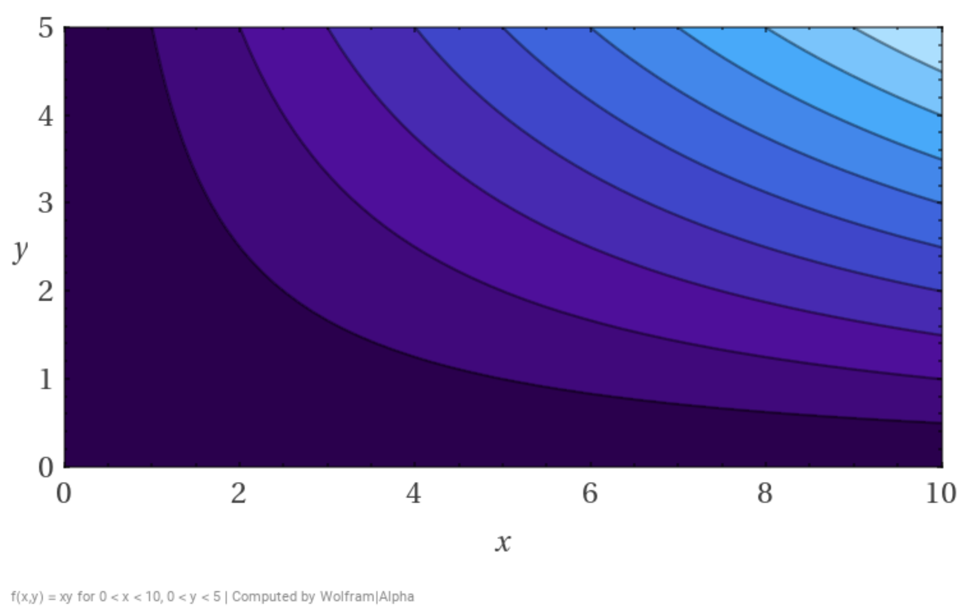
In the following, we will define some properties for functions over lattices and partially ordered sets. For your first quarter of micro, you can always consider $X \subset \mathbb{R}^K$ and $T \subset \mathbb{R}^N$ for some $K, N \in \mathbb{N}$ with our usual partial order on these sets.

Definition 1.2 (Supermodularity)**.** *Let $f : X \times T \to \mathbb{R}$ for some lattice $(X, \preceq)$. Then we say that f is supermodular in x if*

$$\forall x, x' \in X \; \forall t \in T: \quad f(x \vee x', t) + f(x \wedge x', t) \geq f(x, t) + f(x', t)$$

We say that f is submodular in x if instead:

$$\forall x, x' \in X \; \forall t \in T: \quad f(x \vee x', t) + f(x \wedge x', t) \leq f(x, t) + f(x', t)$$



$f(x, y) = xy$ is supermodular in (x, y).

Definition 1.3 (Strict Supermodularity). *Let $f : X \times T \to \mathbb{R}$ for some lattice $(X, \preceq)$. Then we say that f is strictly supermodular in x if*

$$\forall x \neq x' \in X \; \forall t \in T: \quad f(x \vee x', t) + f(x \wedge x', t) > f(x, t) + f(x', t)$$

We say that f is strictly submodular in x if instead:

$$\forall x \neq x' \in X \; \forall t \in T: \quad f(x \vee x', t) + f(x \wedge x', t) < f(x, t) + f(x', t)$$

Definition 1.4 (log-Supermodularity). *Let $f : X \times T \to \mathbb{R}$ for some lattice $(X, \preceq)$. Then we say that f is log-supermodular in x if $g(x,t) = \log(f(x,t))$ is supermodular in x. Define log-submodularity analogously.*

Definition 1.5 (Increasing Differences). *Let $f : X \times T \to \mathbb{R}$ for some lattice $(X, \preceq_X)$ and some partially ordered set $(T, \preceq_T)$. Then we say that f has increasing differences (ID) in x and t if*

$$\forall x \preceq_X x' \in X \; \forall t \preceq_T t' \in T: \quad f(x', t') - f(x, t') \geq f(x', t) - f(x, t)$$

or equivalently

$$\forall x, x' \in X \; \forall t \preceq_T t' \in T: \quad f(x' \vee x, t') - f(x' \vee x, t) \geq f(x', t') - f(x', t)$$

Definition 1.6 (Strictly Increasing Differences). *Let $f : X \times T \to \mathbb{R}$ for some lattice $(X, \preceq_X)$ and some partially ordered set $(T, \preceq_T)$. Then we say that f has increasing differences (ID) in x and t if*

$$\forall x \preceq_X x' \in X \text{ with } x \neq x' \; \forall t \preceq_T t' \in T: \quad f(x', t') - f(x, t') > f(x', t) - f(x, t)$$

or equivalently

$$\forall x \neq x' \in X \; \forall t \preceq_T t' \in T: \quad f(x' \vee x, t') - f(x' \vee x, t) > f(x', t') - f(x', t)$$

Example 1.1 (Increasing Differences and Supermodularity). *Let $f : X \times T \to \mathbb{R}$ for some suitable X and T. Then supermodularity in x plus increasing differences in x and t is weaker than supermodularity in (x,t). As an illustration, assume that $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Then supermodularity in x plus increasing differences in x and t requires that:*

$$sgn\left(\left[\begin{array}{cc|cc} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial t_1} & \frac{\partial^2 f}{\partial x_1 \partial t_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial t_1} & \frac{\partial^2 f}{\partial x_2 \partial t_2} \\ \hline \frac{\partial^2 f}{\partial t_1 \partial x_1} & \frac{\partial^2 f}{\partial t_1 \partial x_2} & \frac{\partial^2 f}{\partial t_1^2} & \frac{\partial^2 f}{\partial t_1 \partial t_2} \\ \frac{\partial^2 f}{\partial t_2 \partial x_1} & \frac{\partial^2 f}{\partial t_2 \partial x_2} & \frac{\partial^2 f}{\partial t_2 \partial t_1} & \frac{\partial^2 f}{\partial t_2^2} \end{array}\right]\right) = \left[\begin{array}{cc|cc} \cdot & + & + & + \\ + & \cdot & + & + \\ \hline + & + & \cdot & \cdot \\ + & + & \cdot & \cdot \end{array}\right]$$

whereas supermodularity in (x,t) would imply

$$sgn\left(\left[\begin{array}{cc|cc} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial t_1} & \frac{\partial^2 f}{\partial x_1 \partial t_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial t_1} & \frac{\partial^2 f}{\partial x_2 \partial t_2} \\ \hline \frac{\partial^2 f}{\partial t_1 \partial x_1} & \frac{\partial^2 f}{\partial t_1 \partial x_2} & \frac{\partial^2 f}{\partial t_1^2} & \frac{\partial^2 f}{\partial t_1 \partial t_2} \\ \frac{\partial^2 f}{\partial t_2 \partial x_1} & \frac{\partial^2 f}{\partial t_2 \partial x_2} & \frac{\partial^2 f}{\partial t_2 \partial t_1} & \frac{\partial^2 f}{\partial t_2^2} \end{array}\right]\right) = \left[\begin{array}{cc|cc} \cdot & + & + & + \\ + & \cdot & + & + \\ \hline + & + & \cdot & + \\ + & + & + & \cdot \end{array}\right]$$

If x and t are one-dimensional, they are the same.

Definition 1.7 (Single Crossing). *Let $f : X \times T \to \mathbb{R}$ for some partially ordered sets $(X, \preceq_X)$ and $(T, \preceq_T)$. We say that f fulfills the single crossing property if*

$$\begin{aligned} \forall x \preceq_X x' \in X \; \forall t \preceq_T t' \in T: \quad & f(x', t) - f(x, t) > 0 \implies f(x', t') - f(x, t') > 0 \\ \forall x \preceq_X x' \in X \; \forall t \preceq_T t' \in T: \quad & f(x', t) - f(x, t) \geq 0 \implies f(x', t') - f(x, t') \geq 0 \end{aligned}$$

Definition 1.8 (Strict Single Crossing). *We say that f fulfills the strict single crossing property if, additionally*

$$\forall x \preceq_X x' \in X \text{ with } x \neq x' \; \forall t \preceq_T t' \in T: \quad f(x', t) - f(x, t) \geq 0 \implies f(x', t') - f(x, t') > 0$$

Definition 1.9 (Quasi-Supermodularity). *Let $f : X \times T \to \mathbb{R}$ for some lattice $(X, \preceq)$. Then we say that f is quasi-supermodular in x if*

$$\forall x, x' \in X \; \forall t \in T: \quad f(x, t) \geq f(x \wedge x', t) \implies f(x \vee x', t) \geq f(x', t)$$

We say that f is quasi-submodular in x if instead

$$\forall x, x' \in X \; \forall t \in T: \quad f(x, t) \leq f(x \wedge x', t) \implies f(x \vee x', t) \leq f(x', t)$$

What are the relationships between these properties?

- Do any of them imply any others?
- Which are the weakest among them?
- Are they cardinal or ordinal properties?

Why are we doing all this? $\implies$ **To use Theorems like the following:**

Definition 1.10 (Strong Set Order)**.** *Let $X, Y \subset \mathbb{R}$ be two sets of real numbers. We say that X dominates Y in the strong set order, denoted by $X \geq_{SSO} Y$ if*

$$\forall a \in X \, \forall b \in Y \qquad \max\{a, b\} \in X \wedge \min\{a, b\} \in Y$$

Theorem 1.1 (Topkis' Theorem)**.** *Let $(X, \preceq)$ be a lattice and $f : X \times T \to \mathbb{R}$. If f has increasing differences in (x, t) and is supermodular in x, then*

$$X^*(t) = \arg\max_{x \in X} f(x, t)$$

is increasing in the Strong Set Order.

That means that using these really "weak" properties, we can predict a lot about the choices of a utility maximizer. For example: how does somebody with a supermodular utility function act if we change (for example) the price of a good?

2 Derivatives

Theorem 2.1 (Chain Rule)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Define $h : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R} : \; h(x) = f(g(x))$.*
Then h', the derivative of h, can be calculated as:

$$h'(x) = f'(g(x)) \, g'(x)$$

Theorem 2.2 (Product Rule)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Define $h : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R} : \; h(x) = f(x)g(x)$.*
Then h', the derivative of h, can be calculated as:

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

Theorem 2.3 (Quotient Rule)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Define $h : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R} : \; h(x) = \frac{f(x)}{g(x)}$. Then, if $g'(x) \neq 0$, h', the derivative of h, can be calculated as:*

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g'(x))^2}$$

Theorem 2.4 (Leibniz-Rule)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous in x and t. Let its partial derivative $\frac{\partial f}{\partial t}$ also be continuous in x and t. Let $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ be two continuously differentiable functions. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a(x)}^{b(x)} f(x, t)\mathrm{d}t\right) = f(x, b(x))\frac{\mathrm{d}}{\mathrm{d}x}b(x) - f(x, a(x))\frac{\mathrm{d}}{\mathrm{d}x}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t)\mathrm{d}t$$