Handout 11 - ECON703 (Fall 2023)

1 Midterm - Common Pitfalls

Assuming Interior Solutions:
Assume we face the following optimization problem:

$$\max_{x \in \mathbb{R}^n_{\geq 0}} u(x) \quad \text{s.t.} \quad x_i \geq 0 \quad \forall i = 1, \dots, n, \quad p'x \leq y$$

We can **NOT** automatically jump to equating marginal utilities!

$$\frac{\text{MU}_i}{p_i} = \frac{\text{MU}_j}{p_j} \quad \text{only applies if we consume both goods in optimum}$$

If we do not consume good i, then $\frac{\text{MU}_i}{p_i} \leq \frac{\text{MU}_j}{p_j} \quad \forall j$

Forgetting Lagrange-Multipliers:
Similarly, if we set up a Lagrangian for an optimization problem, we need a Lagrange multiplier for **every** constraint. For the problem used in the previous paragraph:

$$\mathcal{L} = u(x) + \sum_{i=1}^{n} \lambda_i x_i + \lambda_y \left(y - \sum_{i=1}^{n} p_i x_i \right)$$

If we can somehow deduce that individual constraints will not bind, we can get rid of those multipliers, but we have to be careful.

Not checking whether a solution makes sense / is well-defined:
When you derive a solution, you should check whether it is well-defined (for example, it doesn't contain a square root of a negative number) and makes sense in the economic context (for example, no negative demand for a consumer good).

2 Midterm - Cost-Function Problem

Consider the following two production technologies:

$$\text{Technology 1:} \quad \frac{1}{Q_1} = \frac{1}{K_1} + \frac{1}{L_1} \quad \text{and} \quad \text{Technology 2:} \quad \sqrt{Q_2} = \sqrt{K_2} + \sqrt{L_2}$$

We are interested in the cost function of a producer with access to both technologies. The producer buys labor at a price $w > 0$ and rents capital at a price $r > 0$.

In other words, we are interested in the solution of the following minimization problem.

$$\begin{aligned} c(Q; w, r) = &\min_{K_1, K_2, L_1, L_2} w(L_1 + L_2) + r(K_1 + K_2) \\ &\text{s.t.} \quad K_1 \geq 0, \ K_2 \geq 0, \ L_1 \geq 0, \ L_2 \geq 0, \quad Q_1 + Q_2 = Q \\ &\frac{1}{Q_1} = \frac{1}{K_1} + \frac{1}{L_1}, \quad \sqrt{Q_2} = \sqrt{K_2} + \sqrt{L_2} \end{aligned}$$

Now, we could throw KKT at this, but it would be quite tedious. Instead, we can derive the cost functions for both technologies individually and make use of the fact that they are **Constant Returns to Scale (CRS)**.

$$c_1(Q_1; w, r) = \min_{K_1, L_1} wL_1 + rK_1 \quad \text{s.t.} \quad K_1 \geq 0,\ L_1 \geq 0, \quad \frac{1}{Q_1} = \frac{1}{K_1} + \frac{1}{L_1}$$
$$c_2(Q_2; w, r) = \min_{K_2, L_2} wL_2 + rK_2 \quad \text{s.t.} \quad K_2 \geq 0,\ L_2 \geq 0, \quad \sqrt{Q_2} = \sqrt{K_2} + \sqrt{L_2}$$

Since the technologies are **CRS**, the cost functions will take the form:

$$c_1(Q_1; w, r) = Q_1 c_1(1; w, r) \quad \text{and} \quad c_2(Q_2; w, r) = Q_2 c_2(1; w, r)$$

where $c_1(1; w, r) > 0$ and $c_2(1; w, r) > 0$ are constants that depend on the prices alone. As you can see, the marginal cost of producing an additional unit of quantity is constant for both technologies. Thus, the only scenario where it can be rational to use both technologies is if $c_1(1; w, r) = c_2(1; w, r)$. Otherwise, we will only use the technology associated with the smaller constant.

Therefore, we get the following relationship:

$$c(Q; w, r) = Q \min\{c_1(1; w, r), c_2(1; w, r)\}$$

Let's calculate $c_1(1; w, r)$ and $c_2(1; w, r)$. Let $\rho_1 = -1$ and $\rho_2 = \frac{1}{2}$ and consider the following optimization problem.

$$\begin{aligned} c_i(Q_1; w, r) &= \min_{K_i, L_i} wL_i + rK_i \quad \text{s.t.} \quad K_i \geq 0,\ L_i \geq 0, \quad Q_i^{\rho_i} = K_i^{\rho_i} + L_i^{\rho_i} \\ &= \min_{K_i, L_i} wL_i + rK_i \quad \text{s.t.} \quad K_i \geq 0,\ L_i \geq 0, \quad Q_i = (K_i^{\rho_i} + L_i^{\rho_i})^{\frac{1}{\rho_i}} \end{aligned}$$

Let $\sigma_i = \frac{1}{1-\rho_i}$ and thus $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = 2$. Following John's arguments from `https://users.ssc.wisc.edu/~jkennan/teaching/CEScostfunction.pdf`, we obtain the following relationship.

$$c_i(1; w, r)^{1-\sigma_i} = w^{1-\sigma_i} + r^{1-\sigma_i}$$

and thus

$$c_1(1; w, r) = \left(w^{\frac{1}{2}} + r^{\frac{1}{2}}\right)^2 \quad \text{and} \quad c_2(1; w, r) = \left(w^{-1} + r^{-1}\right)^{-1} = \frac{wr}{w+r}$$

Now, we need to figure out which one is smaller.

$$\left(w^{\frac{1}{2}} + r^{\frac{1}{2}}\right)^2 > \frac{wr}{w+r} \iff \left(w^{\frac{1}{2}} + r^{\frac{1}{2}}\right)^2 - \frac{wr}{w+r} > 0 \iff \frac{r^2 + rw + w^2}{r+w} + 2\sqrt{rw} > 0$$

Which clearly holds if $w > 0$ and $r > 0$. Thus, the producer will only ever use the second technology. Therefore, the producer's cost function is $c(Q; w, r) = \frac{wr}{w+r}$.

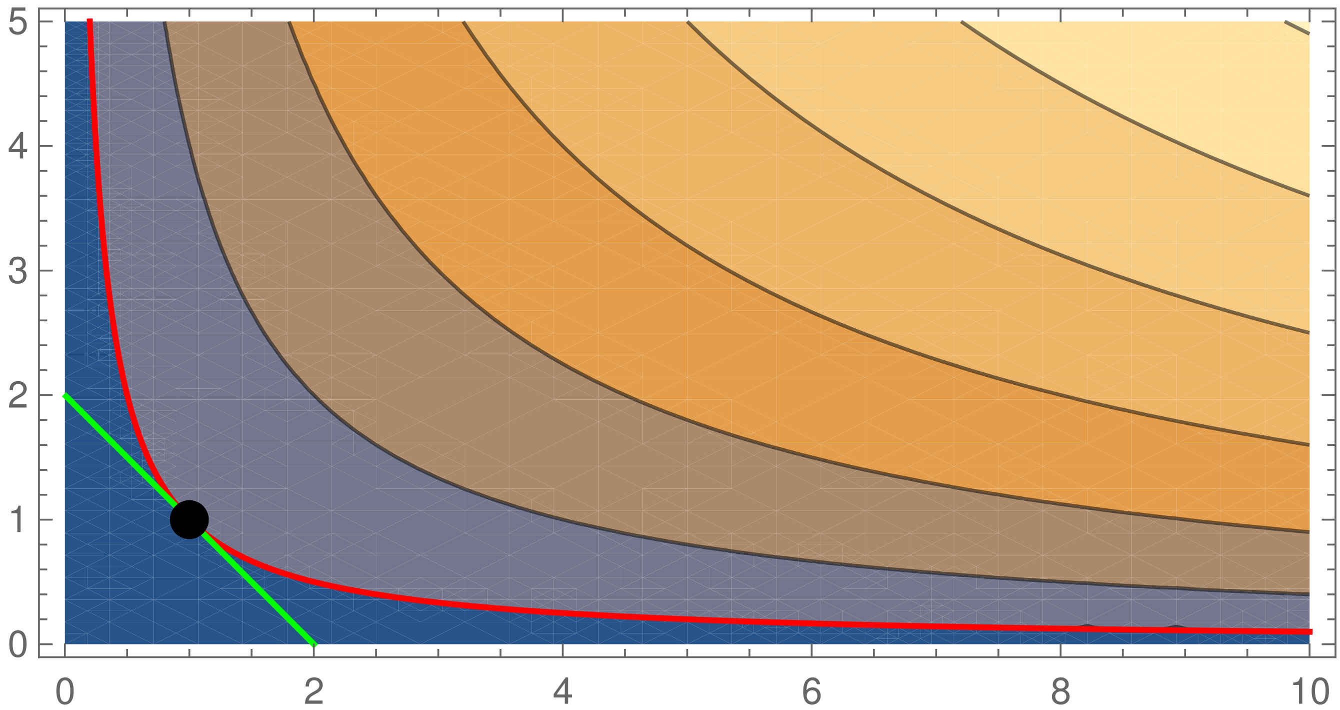
3 Implicit Function Theorem

Theorem 3.1 (Implicit Function Theorem on Euclidean Spaces)**.** *Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a continuously differentiable function. Let $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+m}$ denote a point such that $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. Let $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{n+m}$ be a point such that $f(\mathbf{a}, \mathbf{b}) = 0$. Let the Jacobian of f be denoted as follows*

$$J_f(\mathbf{a}, \mathbf{b}) = \left[\begin{array}{ccc|ccc} \frac{\partial f_1}{\partial x_1}(\mathbf{a}, \mathbf{b}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}, \mathbf{b}) & \frac{\partial f_1}{\partial y_1}(\mathbf{a}, \mathbf{b}) & \dots & \frac{\partial f_1}{\partial y_m}(\mathbf{a}, \mathbf{b}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}, \mathbf{b}) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}, \mathbf{b}) & \frac{\partial f_m}{\partial y_1}(\mathbf{a}, \mathbf{b}) & \dots & \frac{\partial f_m}{\partial y_m}(\mathbf{a}, \mathbf{b}) \end{array}\right] = [X(\mathbf{a}, \mathbf{b}) | Y(\mathbf{a}, \mathbf{b})]$$

If $Y(\mathbf{a}, \mathbf{b})$ is invertible then there are some open set $U \subset \mathbb{R}^n$ with $\mathbf{a} \in U$ and a unique continuously differentiable function $g : U \to \mathbb{R}^m$ such that $g(\mathbf{a}) = b$ and $\forall \mathbf{x} \in U :\ f(\mathbf{x}, g(\mathbf{x})) = 0$. Moreover, the Jacobian of g for $\mathbf{x} \in U$ is given by:

$$J_g(\mathbf{x}) = -\left[Y(\mathbf{x}, g(\mathbf{x}))\right]^{-1} X(\mathbf{x}, g(\mathbf{x}))$$



Example 3.1 (Iso-Utility Curves of a Cobb-Douglas Function). *Consider* $f : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ *given by* $f(x, y) = \sqrt{xy} - 1$. *This is a Cobb-Douglas utility function shifted to fit the implicit function theorem. Take the point* $(x, y) = (1, 1)$, *then* $f(1, 1) = 0$.

The Jacobian of f *is*

$$J_f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \end{bmatrix}$$

Using the implicit function theorem, we can calculate the gradient of the iso-utility curve at $(1, 1)$. *Let* $g(x)$ *be such that* $f(x, g(x)) = 0$. *In other words, let* $g(x)$ *describe the iso-utility curve for the value 1.*

Then

$$J_g(x)\Big|_{x=1,y=1} = -\left(\frac{\sqrt{x}}{2\sqrt{y}}\right)^{-1} \frac{\sqrt{y}}{2\sqrt{x}}\Big|_{x=1,y=1} = -\frac{y}{x}\Big|_{x=1,y=1} = -1$$

This shows that the slope of the indifference curve through $(1, 1)$ *at* $(1, 1)$ *is* -1.