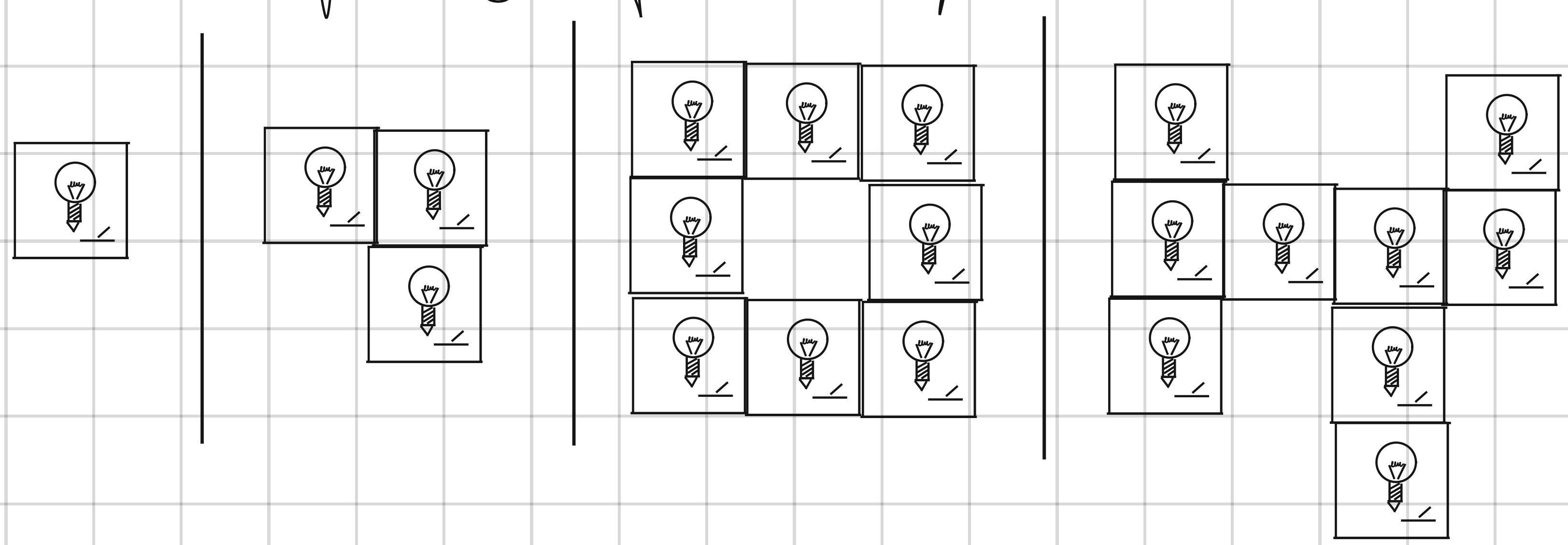
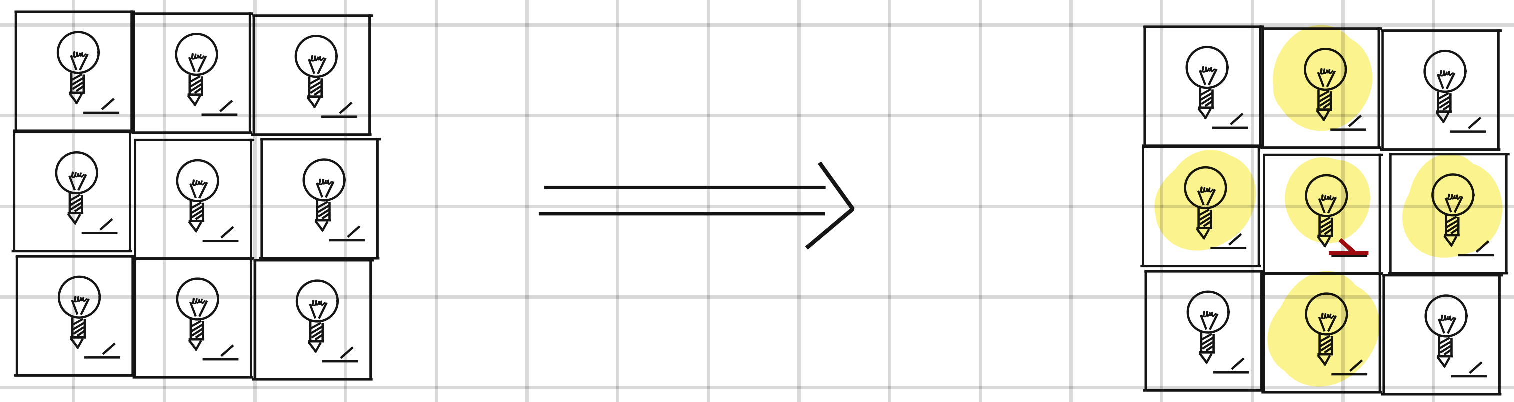
The Lamp-Game

We are considering the following game:
Imagine a set of boxes, each containing a lamp and a switch. The boxes are arranged on a grid, so that each box shares at least one wall with another box (if there are two or more boxes)

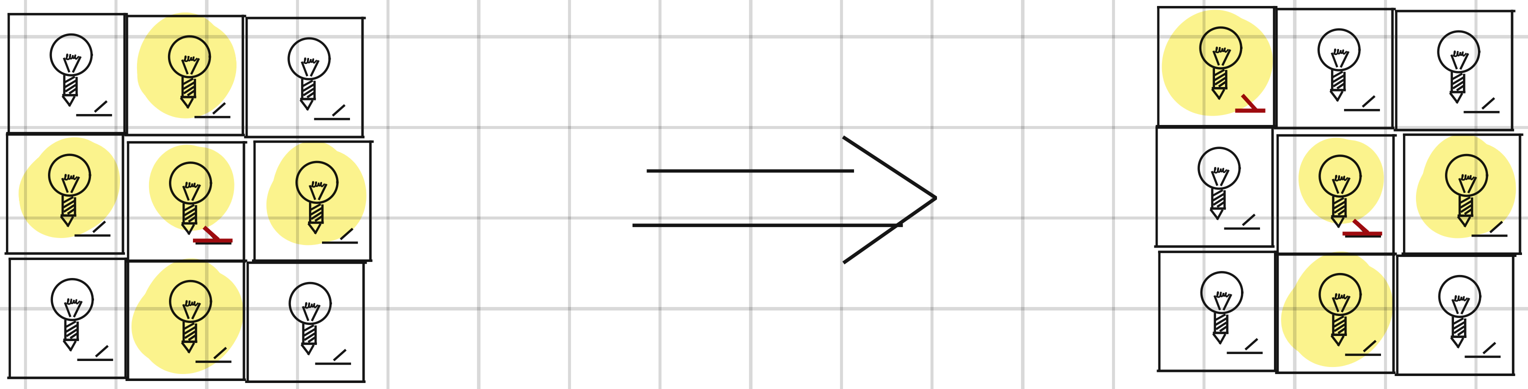
Legal box arrangements for this game could look like the following, for example:



Each switch controls the lamp in its own box and the lamps in boxes that share a wall with its box. Thus, if we activate the switch in the central box, the following occurs:



If additionally, we use the switch in the top left, multiple lamps change their state.

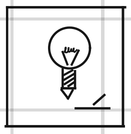


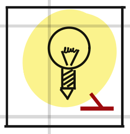
The goal of the game is to activate all lamps.

$\Longrightarrow$ Is this game solvable for any legal arrangement of boxes?

Yes!

Proof: Let X be the set of all boxes in a game. We will prove the solvability by induction.

Base Case:  is solvable:

 is the solution.

Induction Assumption: Any game set-up consisting of n boxes is solvable.

Induction Step: Consider a game setup consisting of $n+1$ boxes. And let X be the set of all boxes. Call a solution of the game setup without the box $Q \in X$ a Q-solution of the game.
$\forall Q \in X$, there is a Q-solution by induction assumption.

We use a <u>case distinction</u>:

1. $\exists Q' \in X$ such that the Q-solution for Q' solves the full game, i.e. the setup including Q' consisting of $n+1$ boxes.

$\Longrightarrow$ if so we are done.

2. $\forall Q \in X$, the Q-solution does not solve the full game with $n+1$ boxes.

2a: <u>$n+1$ is even</u>

if $n+1$ is even, we can apply all Q-solutions for $Q \in X$. Every Q solution changes the state of every lamp <u>but</u> Q.
Thus the state of each lamp changes an odd number of times. In the end, each lamp is therefore on.

2b: <u>$n+1$ is odd</u>

if $n+1$ is odd, there is at least one box say Q' with an even number of neighbors.
Let U be the set containing Q' and its neighbors. Then $|U|$ is odd and $|X \setminus U|$ is even.

Now, use the switch in Q'. All lamps in U are now on. Then apply the Q-solution for all $Q \in X \setminus U$. Since each Q-solution changes the state of all lamps but the one in the corresponding Q, the following happens:

$\forall Q \in U$: every lamp started in the on-state and switches states an even number of times.

$\Rightarrow$ All lamps in U are on in the end.

$\forall Q \in X \setminus U$: every lamp started in the off-state and switches states an odd number of times, namely for every Q-solution in $X \setminus U$ but its own.

$\Rightarrow$ All lamps in $X \setminus U$ are on in the end

Thus we have found a solution.

Every legal arrangement is solvable. $\square$